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A NEW TYPE OF FUZZY COVERING PROPERTY FOR FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and study a new type of fuzzy S-closedness in a fuzzy topological space with respect to certain fuzzy grill $\mathcal{G}$ and through some specific α-shading ($0 < \alpha < 1$), named as fuzzy $\mathcal{G}_\alpha^S$-closedness. The work done in this paper aspires to get, in a new perspective, certain analogues of the results and concepts-usually encountered in connection with the study of S-closedness in topological and fuzzy topological spaces.

1. Introduction and Preliminaries:

The introductions of fuzzy sets by Zadeh [21] in 1965 and fuzzy topology by Chang [5] in 1968, created a new area for the subsequent rapid development for the extensions of various concepts and their properties from classical set topological case to the wider framework of fuzzy topological space. Among them, an interesting topological covering property named S-closedness was introduced by Thompson [20] and in its hierarchy, fuzzy S-closedness was first introduced by Mukherjee and Ghosh [17] in 1989. It was Gantner et al.[8] who designed a totally novel concept of covering named α-shading (α-being a member of a designated lattice) and developed a new definition of fuzzy compactness.

On the other hand, observing the importance and wide applicabilities of the idea of grill in set topology, the notion of fuzzy grill was innovated by Azad [1] in 1981.

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Keeping pace with the recent trend, the ideas of fuzzy grill and α-shading were mixed in [6] for the introduction of fuzzy $\mathcal{G}_\alpha$-compactness and fuzzy $\mathcal{G}_\alpha$-almost compactness.

In this paper, our intention is to define and study S-closedness using both α-shading and fuzzy grill. For developing this idea, in Section 2, we define semi-open α-shading, proximate $\mathcal{G}_\alpha$-shading and then fuzzy $\mathcal{G}_\alpha^S$-closedness. Inter-relations among fuzzy $\mathcal{G}_\alpha$-compactness, fuzzy $\mathcal{G}_\alpha$-almost compactness and fuzzy $\mathcal{G}_\alpha^S$-closedness are also found in this section.
In Section 3, different characterizations of $\mathcal{G}_\alpha^S$-closedness along with certain associated results are obtained by using a number of fuzzy topological tools like fuzzy grill, prefilter, prefilterbases etc.
The last section i.e., Section 4 is intended for the study of $\mathcal{G}_\alpha^S$-closedness under different types of functions to ensure how $\mathcal{G}_\alpha^S$-closedness of a fuzzy topological space may be transferred to another space.

According to Zadeh [21], a fuzzy set in a non-empty set X is defined to be a function from X into the closed unit interval $[0, 1]$. The basic fuzzy sets in X taking on respectively the constant values 0 and 1 at each point of X are denoted by 0_X and 1_X respectively.
The complement $\overline{A}$ of a fuzzy set A is denoted by $(1_X - A)$. A fuzzy topological space [henceforth fts, for short] (X, τ) is defined to be a non-null set X equipped with some fuzzy topology τ [as given by Chang [5]], I^X denotes the set of all fuzzy sets in a non-empty set X, where I stands for $[0, 1]$. The set $\{x \in X : A(x) > 0\}$ is called the support of a fuzzy set A in X.
A fuzzy point in X is a fuzzy set in X with a singleton support $\{x\}$ (say) and value $\lambda(0 < \lambda \leq 1)$ and is denoted by x_λ.
For two fuzzy sets A, B in X,

(i) $A \leq B$ means $A(x) \leq B(x), \forall\, x \in X$, the negation of which is denoted by $A \not\leq B$.

(ii) $A = B$ means $A(x) = B(x), \forall\, x \in X$.

(iii) A fuzzy point x_λ is said to be contained in A if $x_\lambda \leq A$ i.e. $\lambda \leq A(x)$.

(iv) A is said to be quasi-coincident [14] with B written as AqB if $A(x) + B(x) > 1$ for some $x \in X$, the negation of which is denoted by $A\bar{q}B$.

(v) B is said to be q-nbd of A, if there is a fuzzy open set U in X such that $AqU \leq B$. If in addition, B is fuzzy open then B is said to be a fuzzy open q-nbd of A.

The set of all fuzzy open q-nbds of a fuzzy point x_λ is denoted by $Q(x_\lambda)$.

For a family $\{A_\lambda : \lambda \in \Lambda\}$ of fuzzy sets in X, (where Λ is an index set) the union $\bigvee_{\lambda\in\Lambda} A_\lambda$ and the intersection $\bigwedge_{\lambda\in\Lambda} A_\lambda$ of the collection are the fuzzy sets, defined [21] as: $(\bigvee_{\lambda\in\Lambda} A_\lambda)(x) = \sup\{A_\lambda(x) : \lambda \in \Lambda\},\ x \in X$ and $(\bigwedge_{\lambda\in\Lambda} A_\lambda)(x) = \inf\{A_\lambda(x) : \lambda \in \Lambda\},\ x \in X$.

Also as shown in [21], $\overline{\bigvee_{\lambda\in\Lambda} A_\lambda} = \bigwedge_{\lambda\in\Lambda} \overline{A_\lambda}$ and $\overline{\bigwedge_{\lambda\in\Lambda} A_\lambda} = \bigvee_{\lambda\in\Lambda} \overline{A_\lambda}$.

Before we proceed further, we need the following well known definitions and results:

Definition 1.1 ([2]). *For a fuzzy set A in an fts (X, τ), $int(A)$ and $cl(A)$ are defined by*

$int(A) = \bigvee\{B : B \leq A \text{ and } \mathrm{B} \in \tau\}$ *and* $cl(A) = \bigwedge\{B : A \leq B \text{ and } (1_\mathrm{X} - \mathrm{B}) \in \tau\}$.

Result 1.2 ([2]). *For any fuzzy sets A, $A_\lambda(\lambda \in \Lambda)$ in an fts X,*

(i) $1 - int(A) = cl(1 - A)$; $1 - cl(A) = int(1 - A)$ and A is fuzzy open (closed) iff $A = int(A)$ [resp. $A = cl(A)$].

(ii) $\bigvee_{\lambda\in\Lambda} cl(A_\lambda) \leq cl(\bigvee_{\lambda\in\Lambda} A_\lambda)$, equality holds when Λ is finite, and $\bigvee_{\lambda\in\Lambda} int(A_\lambda) \leq int(\bigvee_{\lambda\in\Lambda} A_\lambda)$.

Definition 1.3 ([2]). *A fuzzy set A in an fts (X, τ) is said to be*

(i) fuzzy semi-open if there is a B in τ such that $B \leq A \leq cl(B)$,

(ii) fuzzy semi-closed if $\exists$ a fuzzy closed set C in X for which $int(C) \leq A \leq C$.

(iii) fuzzy regular open if $int(cl(A)) = A$.

(iv) fuzzy regular closed if $cl(int(A)) = A$.

Result 1.4 ([2]). *Let A be any fuzzy set in an fts (X, τ). Then the following are equivalent:*

(i) A is fuzzy semi-closed,

(ii) $\overline{A}$ is fuzzy semi-open,

(iii) $int(cl(A)) \leq A$,

(iv) $cl(int(\overline{A})) \geq \overline{A}$.

Definition 1.5 ([17]). *For any fuzzy set A in an fts (X, τ), $Scl(A)$ is the intersection of all fuzzy semi-closed sets containing A.*

Corollary 1.6 ([17]). *A fuzzy set A in (X, τ) is fuzzy semi-closed iff $A = Scl(A)$.*

Definition 1.7 ([21]). *Let $f : X \to Y$ be a function and let A and B be two fuzzy sets in X and Y respectively. Then $f(A)$ is a fuzzy set in Y defined by*

$$f(A)(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} A(z), & \text{when } \mathrm{f^{-1}(y)} \neq \phi \\ 0, & \text{when } \mathrm{f^{-1}(y)} = \phi \end{cases}$$

and $f^{-1}(B)$ is a fuzzy set in X defined by $f^{-1}(B)(x) = B(f(x))$, for $x \in X$.

Result 1.8 ([5]). *Let $f : X \to Y$ be a function. Then*

(i) $f^{-1}(1 - B) = 1 - f^{-1}(B)$, for any fuzzy set B in Y.

(ii) $B_1 \leq B_2 \Rightarrow f^{-1}(B_1) \leq f^{-1}(B_2)$, where B_1, B_2 are fuzzy sets in Y.

(iii) $A_1 \leq A_2 \Rightarrow f(A_1) \leq f(A_2)$, where A_1, A_2 are fuzzy sets in X.

(iv) $B \geq ff^{-1}(B)$, equality occurs if f is surjective, for any $B \in I^Y$.

(v) $A \leq f^{-1}f(A)$, equality occurs if f is injective, where $A \in I^X$.

(vi) $f(1 - A) \geq 1 - f(A)$, equality occurs if f is bijective, where $A \in I^X$.

Definition 1.9 ([2]). *Let $\mathcal{G}$ be a non-null collection of fuzzy sets in an fts (X, τ) such that*
(i) $0_X \notin \mathcal{G}$,
(ii) $A \in \mathcal{G}$, $B \in I^X$ and $A \leq B \Rightarrow B \in \mathcal{G}$ and
(iii) $A, B \in I^X$ and $A \vee B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.
Then $\mathcal{G}$ is called a fuzzy grill on X.

Throughout this paper, we call an fts (X, τ) endowed with a fuzzy grill $\mathcal{G}$ a fuzzy $\mathcal{G}$-space to be denoted by $(X, \tau, \mathcal{G})$; also Λ, appearing anywhere, is an index set, and α appearing everywhere, will be assumed to satisfy $0 < \alpha < 1$.

2. Fuzzy $\mathcal{G}_\alpha^S$-closedness

In this section our main concern is to introduce a new kind of fuzzy covering property for a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ in terms of a fuzzy grill $\mathcal{G}$ on X and the idea of α-shading, where α satisfies (as already mentioned) $0 < \alpha < 1$. To define $\mathcal{G}_\alpha^S$-closedness we require the following:

Definition 2.1 ([8]). *Let (X, τ) be any fts and $0 < \alpha < 1$.*
(i) A collection $\mathcal{U}$ of fuzzy sets in X is said to be an α-shading of X if for each $x \in X$, there exists some $U \in \mathcal{U}$ such that $U(x) > \alpha$.
In particular, if each member of $\mathcal{U}$ is fuzzy open then $\mathcal{U}$ is called an open α-shading of X.
(ii) A subcollection $\mathcal{U}_1$ of $\mathcal{U}$ is called an α-subshading of $\mathcal{U}$ if $\mathcal{U}_1$ is itself an α-shading of X.
(iii) (X, τ) is called α-compact space if each open α-shading of X has a finite α-subshading.

Definition 2.2 ([6]). *Let $\mathcal{U}$ be any open α-shading of X in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ and $0 < \alpha < 1$. A finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ is said to be a $\mathcal{G}_\alpha$-shading of X if $1 - \vee \mathcal{U}_0 \notin \mathcal{G}$.*

Definition 2.3 ([6]). *Let $0 < \alpha < 1$. A fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is called*
(i) fuzzy $\mathcal{G}_\alpha$-compact if each open α-shading of X has a $\mathcal{G}_\alpha$-shading.
(ii) fuzzy $\mathcal{G}_\alpha$-almost compact if for any open α-shading of X, there is a finite subset Λ_0 of Λ for which $1 - \bigvee_{\lambda \in \Lambda_0} cl(U_\lambda) \notin \mathcal{G}$.

In [6], it is already shown that fuzzy $\mathcal{G}_\alpha$-compactness always implies fuzzy $\mathcal{G}_\alpha$-almost compactness but the converse is not true in general.
Let us now define as follows:

Definition 2.4. *Let $(X, \tau, \mathcal{G})$ be a fuzzy $\mathcal{G}$-space and $0 < \alpha < 1$.*
(i) A collection $\mathcal{U}$ of fuzzy semi-open sets is said to be a semi-open α-shading of X if for each $x \in X$, there exists some $U \in \mathcal{U}$ such that $U(x) > \alpha$.
(ii) A finite subcollection $\mathcal{U}_0$ of a semi-open α-shading $\mathcal{U}$ of X is called a proximate $\mathcal{G}_\alpha$-shading of X if $1 - \bigvee_{U \in \mathcal{U}_0} clU \notin \mathcal{G}$.

Definition 2.5. *A fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is said to be fuzzy α-S-closed via the fuzzy grill $\mathcal{G}$ or simply fuzzy $\mathcal{G}_\alpha^S$-closed(where $0 < \alpha < 1$), if each semi-open α-shading of X has a proximate $\mathcal{G}_\alpha$-shading.*

Example 2.6. *Let us take a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, where $X = \{a, b\}$ and $\tau = \{0_X, 1_X\} \vee \{A_n : n = 0, 1, 2, ...\}$, in which for each $n = 0, 1, 2, ...$, $A_n(a) = \frac{n+1}{n+2}$ and $A_n(b) = \frac{n+2}{n+3}$; and $\mathcal{G} = \{G \in I^X / 0 < G(x) \leq 1 : x \in X\}$. If we take $\alpha = 0.3$, then we can easily check that the family $\mathcal{A} = \{A_n : n = 0, 1, 2...\}$ is a semi-open α-shading of X. Now for each $n = 0, 1, 2, ...$, $A_n \not\leq 1 - A_n$. So $clA_n = 1_X$, for each $n = 0, 1, 2, ...$ and hence for each subcollection $\mathcal{A}_0$ of $\mathcal{A}$, $\mathcal{A}_0$ is a proximate $\mathcal{G}_\alpha$-shading of X, since*

$1 - \bigvee_{A \in \mathcal{A}_0} cl(A) = 0_X \notin \mathcal{G}$. *Similarly, each semi-open* α*-shading other than* $\mathcal{A}$ *of* X*, has a proximate* $\mathcal{G}_\alpha$*-shading, since closures of those fuzzy semi-open sets are also* 1_X*. Hence* $(X, \tau, \mathcal{G})$ *is fuzzy* $\mathcal{G}_\alpha^S$*-closed.*

Remark 2.7. *It is clear that a fuzzy* $\mathcal{G}_\alpha^S$*-closed space is fuzzy* $\mathcal{G}_\alpha$*-almost compact. But even a fuzzy* $\mathcal{G}_\alpha$*-compact space may fail to be fuzzy* $\mathcal{G}_\alpha^S$*-closed, which is shown by the following example:*

Example 2.8. *Let* $X = \{a, b\}$ *and* $\tau = \{0_X, 1_X, C\} \vee \{A_n, B_n : n = 4, 5, 6, ...\}$*, in which for each* $n = 4, 5, 6, ...$*,* $A_n(a) = \frac{1}{3}$*,* $A_n(b) = 1 - \frac{1}{n}$*,* $B_n(a) = 0$*,* $B_n(b) = \frac{1}{n}$ *and* $C(a) = \frac{1}{3}$*,* $C(b) = 1$*.*
Then τ *is a fuzzy topology on* X*.*
Now $\mathcal{G} = \{G \in I^X / 0 \le G(a) \le 1, 0 < G(b) \le 1\}$ *is a fuzzy grill on* X*. Thus* $(X, \tau, \mathcal{G})$ *is a fuzzy* $\mathcal{G}$*-space.*
It is easy to check that $cl(A_n) = 1 - B_n$ *for each* $n = 4, 5, 6, 7, ...$ *Let us take* $\alpha = 0.1$*. Then* $\mathcal{B} = \{1 - B_n : n = 4, 5, 6, ...\}$ *is a semi-open* α*-shading of* X*. But we find that for each finite subfamily* $\{1 - B_n : n \in F\}$ *(*F *being a finite subset of* $\mathbb{N} \setminus \{1, 2, 3\}$*),* $1 - \bigvee_{n \in F} cl(1 - B_n) = 1 - \bigvee_{n \in F} (1 - B_n) = \bigwedge_{n \in F} B_n \in \mathcal{G}$*. So* X *is not fuzzy* $\mathcal{G}_\alpha^S$*-closed. But* X *is fuzzy* $\mathcal{G}_\alpha$*-compact since each open* α*-shading* $\mathcal{V}$ *of* X *must contain* 1_X *as a member and hence* $\mathcal{V}$ *has a sub-collection* $\mathcal{V}_0$ *such that* $1_X - \vee \mathcal{V}_0 = 0_X \notin \mathcal{G}$*.*

Our Next goal is to ascertain as to under what condition a fuzzy $\mathcal{G}_\alpha^S$-closed space may be fuzzy $\mathcal{G}_\alpha$-compact. For this we need the following definition:

Definition 2.9 ([10])**.** *An fts* (X, τ) *is called fuzzy regular if for each* $U \in \tau$*;* $U = \bigvee_{\alpha \in \Lambda} U_\alpha$*, where for each* $\alpha \in \Lambda$*,* $U_\alpha \in \tau$ *and* $clU_\alpha \le U$*.*

Theorem 2.10. *Let* $(X, \tau, \mathcal{G})$ *be a fuzzy* $\mathcal{G}$*-space. If* X *is fuzzy regular and fuzzy* $\mathcal{G}_\alpha^S$*-closed, then* X *is fuzzy* $\mathcal{G}_\alpha$*-compact.*

Proof. Let $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ be an open α-shading of X. Then for each $x \in X$, there is some $W_{\lambda_x} \in \mathcal{W}$ such that $W_{\lambda_x}(x) > \alpha$. By fuzzy regularity of X, each W_{λ_x} is a union of a family $\mathcal{V}_{\lambda_x} = \{V^\mu_{\lambda_x} : \mu \in J_x\}$ (where J_x is some index set) of fuzzy open sets $V^\mu_{\lambda_x}$ such that $W_{\lambda_x} = \bigvee \mathcal{V}_{\lambda_x}$ and $cl(V^\mu_{\lambda_x}) \leq W_{\lambda_x}$ for each $V^\mu_{\lambda_x} \in \mathcal{V}_{\lambda_x}$. Thus $\{\mathcal{V}_{\lambda_x} : x \in X\}$ is also an open (and hence a semi-open) α-shading of X and as X is fuzzy $\mathcal{G}^S_\alpha$-closed, $\exists$ finite subsets X_0 of X and J^0_x of J_x, $\forall\ x \in X_0$ for which $1 - cl\{\bigvee V^\mu_{\lambda_x} : x \in X_0 \text{ and } \mu \in J^0_x\} \notin \mathcal{G}$. Then $1 - \bigvee\{W_{\lambda_x} : x \in X_0\} \notin \mathcal{G}$, so that X is fuzzy $\mathcal{G}_\alpha$-compact. □

Now we want to establish a relation between fuzzy $\mathcal{G}^S_\alpha$-closedness and fuzzy $\mathcal{G}_\alpha$-almost compactness. For this we require the following definition and lemma.

Definition 2.11 ([9])**.** *An fts (X, τ) is said to be fuzzy extremally disconnected if $clU \in \tau$ whenever $U \in \tau$.*

Lemma 2.12 ([9])**.** *In any fts (X, τ), the following statements are equivalent:*
(i) X is fuzzy extremally disconnected.
(ii) The fuzzy closure of each fuzzy semi-open set is fuzzy open.
(iii) $clU = Scl(U)$ for each fuzzy semi-open set U in X.

Theorem 2.13. *A fuzzy extremally disconnected and fuzzy $\mathcal{G}_\alpha$-almost compact space is fuzzy $\mathcal{G}^S_\alpha$-closed.*

Proof. Let $\mathcal{S}$ be a semi-open α-shading of a fuzzy extremally disconnected fuzzy $\mathcal{G}_\alpha$-almost compact space $(X, \tau, \mathcal{G})$. Then $\{cl(S) : S \in \mathcal{S}\}$ is an open α-shading of X (by Lemma 2.12).
Now since X is fuzzy $\mathcal{G}_\alpha$-almost compact, we can find a finite subfamily $\mathcal{S}_0$ of $\mathcal{S}$ for which $1 - \bigvee_{S \in \mathcal{S}_0} clS \notin \mathcal{G}$ and consequently X is fuzzy $\mathcal{G}^S_\alpha$-closed. □

Remark 2.14. *From what has gone so far, we conclude that if a fuzzy $\mathcal{G}$-space is fuzzy extremally disconnected and fuzzy regular, then the concepts of fuzzy $\mathcal{G}_\alpha$-compactness, fuzzy $\mathcal{G}_\alpha$-almost compactness and fuzzy $\mathcal{G}_\alpha^S$-closedness coincide.*

3. Characterizations of fuzzy $\mathcal{G}_\alpha^S$-closedness

In this section we want to characterize fuzzy $\mathcal{G}_\alpha^S$-closed spaces by some well-known fuzzy topological tools such as some particular type of finite intersection property, fuzzy prefilterbase, fuzzy prefilter, fuzzy grill etc.

Definition 3.1. *In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, a family $\mathcal{F} = \{F_i : i \in \Lambda\}$ of fuzzy sets is said to have finite intersection property via the fuzzy grill $\mathcal{G}$ or simply $\mathcal{G}$-f.i.p., if for each finite subset Λ_0 of Λ, $\bigwedge_{i\in\Lambda_0} F_i \in \mathcal{G}$.*

Theorem 3.2. *The following statements are equivalent in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$.*
(i) $(X, \tau, \mathcal{G})$ is fuzzy $\mathcal{G}_\alpha^S$-closed.
(ii) For every family $\mathcal{F} = \{F_i : i \in \Lambda\}$ of fuzzy semi-closed sets with $\bigwedge\{F_i : i \in \Lambda\}(x) < 1-\alpha$ for each $x \in X$, there exists a finite sub-collection $\{F_{ij} : j = 1, 2, ..., n\}$ of $\mathcal{F}$ such that $\bigwedge_{j=1}^{n} int(F_{ij}) \notin \mathcal{G}$.
(iii) For each α-shading $\mathcal{F} = \{F_i : i \in \Lambda\}$ of X by fuzzy regular closed sets, there exists a finite sub-collection $\{F_{ij} : j = 1, 2, ..., n\}$ of $\mathcal{F}$ such that $1 - \bigvee_{j=1}^{n} F_{ij} \notin \mathcal{G}$.
(iv) For each family $\{U_i : i \in \Lambda\}$ of fuzzy regular open sets having $\mathcal{G}$-f.i.p, $(\bigwedge_{i\in\Lambda} U_i)(x) \leq 1-\alpha$, for some $x \in X$.

Proof. $(i) \Rightarrow (ii)$: Let $\mathcal{F} = \{F_i : i \in \Lambda\}$ be a family of fuzzy semi-closed sets with $(\bigwedge_{i\in\Lambda} F_i)(x) < 1-\alpha$ for each $x \in X$. Then $1 - (\bigwedge_{i\in\Lambda} F_i)(x) > \alpha$ for each $x \in X$. So $\mathcal{F}' = \{(1 - F_i) : i \in \Lambda\}$ is a semi-open α-shading of X and by (i), as X is fuzzy $\mathcal{G}_\alpha^S$-closed, there exists a finite subcollection $\{(1 - F_1), (1 - F_2), ..., (1 - F_n)\}$ of $\mathcal{F}'$ such that $1 - \bigvee_{i=1}^{n} cl(1 - F_i) \notin \mathcal{G}$. Thus $\bigwedge_{i=1}^{n} int(F_i) \notin \mathcal{G}$.

$(ii) \Rightarrow (i)$: Let $\{U_i : i \in \Lambda\}$ be any semi-open α-shading of X. Then $\mathcal{U} = \{(1-U_i) : i \in \Lambda\}$ is a family of fuzzy semi-closed sets with $\bigwedge\{(1-U_i) : i \in \Lambda\}(x) < 1-\alpha$ for each $x \in X$. Then by (ii), there exists a finite subcollection $\{(1-U_{i_1}), (1-U_{i_2}), ..., (1-U_{i_n})\}$ of $\mathcal{U}$ such that $\bigwedge\limits_{j=1}^{n} \{int(1-U_{ij})\} \notin \mathcal{G}$. Therefore, $1-\bigvee\limits_{j=1}^{n} cl(U_{ij}) \notin \mathcal{G}$ and hence $(X, \tau, \mathcal{G})$ is fuzzy $\mathcal{G}_\alpha^S$-closed.

$(i) \Rightarrow (iii)$: This is clear from the fact that each fuzzy regular closed set is fuzzy semi open.

$(iii) \Rightarrow (iv)$: Let $\{U_i : i \in \Lambda\}$ be a family of fuzzy regular open sets in X having $\mathcal{G}$-f.i.p.
We need to prove that $(\bigwedge\limits_{i\in\Lambda} U_i)(x) \geq 1-\alpha$ for some $x \in X$. If not, let $(\bigwedge\limits_{i\in\Lambda} U_i)(y) < 1-\alpha$ for each $y \in X$. Then $\mathcal{U} = \{(1-U_i) : i \in \Lambda\}$ is an α-shading of X by fuzzy regular closed sets and hence by (iii), there exists a finite subcollection $\{(1-U_{i_1}), (1-U_{i_2}), ..., (1-U_{i_n})\}$ of $\mathcal{U}$ such that $1-\bigvee\limits_{j=1}^{n}(1-U_{ij}) \notin \mathcal{G} \Rightarrow \bigwedge\limits_{j=1}^{n} U_{ij} \notin \mathcal{G}$, which contradicts the $\mathcal{G}$-f.i.p. of $\{U_i : i \in \Lambda\}$.

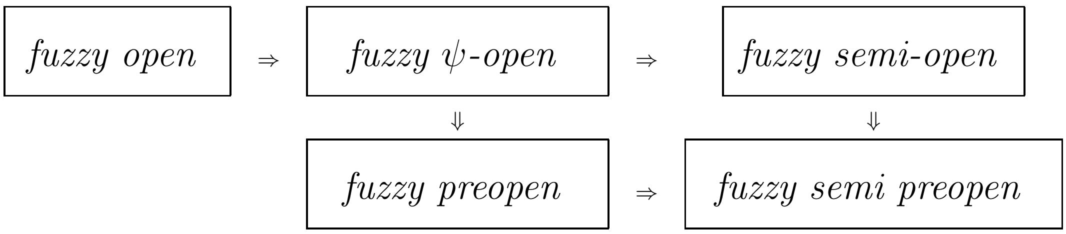
$(iv) \Rightarrow (i)$: Let the condition in (iv) hold and if possible let $(X, \tau, \mathcal{G})$ be not fuzzy $\mathcal{G}_\alpha^S$-closed. Then there exists a semi-open α-shading $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of X such that $1-\bigvee\limits_{\lambda\in\Lambda_0} cl(U_\lambda) \notin \mathcal{G}$, for each finite subset Λ_0 of Λ. So $\bigwedge\limits_{\lambda\in\Lambda_0}(1-cl(U_\lambda)) \in \mathcal{G}$ and hence $\{1-cl(U_\lambda) : \lambda \in \Lambda\}$ is a family of fuzzy regular open sets having $\mathcal{G}$-f.i.p. Then by (iv), $[\bigwedge\limits_{\lambda\in\Lambda}(1-cl(U_\lambda))](x) \geq (1-\alpha)$, for some $x \in X \Rightarrow \bigvee\limits_{\lambda\in\Lambda} clU_\lambda(x) \leq \alpha$, for some $x \in X \Rightarrow \bigvee\limits_{\lambda\in\Lambda} U_\lambda(x) \leq \alpha$, for some $x \in X$, which contradicts the fact that $\mathcal{U}$ is a semi-open α-shading of X. □

There is another way for the study and characterization of fuzzy $\mathcal{G}_\alpha^S$-closedness in terms of the following types of fuzzy sets.

Definition 3.3. *A fuzzy set F in an fts (X,τ) is called*
(i) fuzzy ψ-open [4] *if $F \leq int\, cl\, int F$,*
(ii) fuzzy pre-open [4] *if $F \leq int\, cl F$,*
(iii) fuzzy semi-preopen [19] *if there exists a fuzzy preopen set P such that $P \leq F \leq clP$.*

Note 3.4. *Bin Shahna* [4] *called a ψ-open set as 'α-open set'. Since the Greek alphabet α is being used for a different meaning (i.e., $0 < \alpha < 1$) in this paper, we have used ψ for α in 'α-open set'.*

Remark 3.5. *As is found in* [19]*, the relations of fuzzy open sets with the types of fuzzy open-like sets, considered so far, are given as follows:*



We now want to study fuzzy $\mathcal{G}_\alpha^S$-closedness in terms of the above types of fuzzy open-like sets.

Theorem 3.6. *Let $(X,\tau,\mathcal{G})$ be a fuzzy $\mathcal{G}$-space which is fuzzy $\mathcal{G}_\alpha^S$-closed. Then*
(i) each α-shading of X by fuzzy ψ-open sets has a proximate $\mathcal{G}_\alpha$-shading.
(ii) each α-shading of X by fuzzy pre-open sets has a proximate $\mathcal{G}_\alpha$-shading.

Proof. (i) Follows from the fact that every fuzzy ψ-open set is fuzzy semi-open [by Remark 3.5].

(ii) It can be easily checked that closure of a fuzzy pre-open set is fuzzy semi-open and the rest is immediate. □

Theorem 3.7. *A fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is fuzzy $\mathcal{G}_\alpha^S$-closed iff each α-shading of X by fuzzy semi pre-open sets has a proximate $\mathcal{G}_\alpha$-shading.*

Proof. Let X be fuzzy $\mathcal{G}_\alpha^S$-closed and let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be any α-shading of X by fuzzy semi pre-open sets. By definition, for each $U \in \mathcal{U}$, $\exists$ a fuzzy pre-open set P for which $P \leq U \leq clP \Rightarrow clP \leq clU \leq cl(clP) \Rightarrow clU = clP$ and hence clU is a fuzzy semi-open set [since closure of a fuzzy pre-open set is fuzzy semi-open], for each $U \in \mathcal{U}$. So the collection $\{clU_i : i \in \Lambda\}$ is a family of fuzzy semi-open sets which is also an α-shading of X and hence by fuzzy $\mathcal{G}_\alpha^S$-closedness of X, it has a proximate $\mathcal{G}_\alpha$-shading. This means there exists a finite subset Λ_0 of Λ for which $1 - \bigvee_{i \in \Lambda_0} cl(clU_i) \notin \mathcal{G} \Rightarrow 1 - \bigvee_{i \in \Lambda_0} clU_i \notin \mathcal{G}$. Thus $\mathcal{U}$ has a proximate $\mathcal{G}_\alpha$-shading.

The converse is clear as every fuzzy semi-open set is fuzzy semi pre-open. □

Our next target is to achieve fuzzy $\mathcal{G}_\alpha^S$-closedness via prefilterbases, prefilters and fuzzy grills. To that end we need some concepts as clarified below.

Definition 3.8. *Let (X, τ) be a fuzzy topological space and Ω be the family of all fuzzy closed sets in X.*
(a)[11] *A non-empty family $\mathcal{F}$ of fuzzy sets in X is called a prefilterbase on X, if*
(i) $0_X \notin \mathcal{F}$
(ii) for any $U, V \in \mathcal{F}$, there exists $W \in \mathcal{F}$ such that $W \leq U \wedge V$.
If in addition,
(iii) $A \in \mathcal{F}$ and $A \leq B \in I^X \Rightarrow B \in \mathcal{F}$ holds, then $\mathcal{F}$ is called a prefilter on X.

(b)[3] *A collection $\mathcal{F}$ of fuzzy open sets in X is called an open prefilter on X, if the conditions (i) and (ii) of (a) above hold and in addition*

(iii)$^/$ ($A \in \mathcal{F}$ and $A \leq B \in \tau \Rightarrow B \in \mathcal{F}$) is satisfied.

Similarly a closed prefilter can be defined. Also we can define an open prefilterbase or a closed prefilterbase in a similar manner.
(c) [3] *A non-empty subcollection of Ω (resp. of τ) in a fuzzy topological space (X, τ) is said to form a closed (resp. open) fuzzy grill on X, if*
(i) $0_X \notin \mathcal{G}$
(ii) $A \in \mathcal{G}$ and $C \in \Omega$ (resp. $C \in \tau$) with $A \leq C \Rightarrow C \in \mathcal{G}$ and
(iii) $A, B \in \Omega$ (resp. $A, B \in \tau$) and $A \vee B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 3.9. [6] *For any constant α $(0 < \alpha < 1)$, we define the constant fuzzy set K_α given by $K_\alpha(x) = \alpha$, for all $x \in X$.*

Definition 3.10. *In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, a fuzzy semi-open set U is said to be a semi α-nbd $(0 < \alpha < 1)$ of a crisp point $x \in X$ if $U(x) > \alpha$.*
The collection of all semi α-nbds of x in X is denoted by $\Gamma_\alpha(x)$.

Definition 3.11. *Let (X, τ) be an fts. Then*
(i) a prefilterbase or a prefilter $\mathcal{F}$ on X is said to S_α-adhere at some crisp point $x \in X$, if for each $U \in \Gamma_\alpha(x)$ and for each $F \in \mathcal{F}$, $clUqF$.
(ii) a fuzzy grill $\mathcal{G}$ on X is said to S_α-converge to some crisp point $p \in X$, if corresponding to each $V \in \Gamma_\alpha(p)$, $\exists\ G \in \mathcal{G}$ such that $G \leq clV$.

Theorem 3.12. *In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, the following are equivalent:*
(i) X is fuzzy $\mathcal{G}^S_\alpha$-closed.
(ii) For each family $\{V_\lambda : \lambda \in \Lambda\}$ of fuzzy semi-closed sets such that $\bigwedge_{\lambda \in \Lambda} V_\lambda < K_{1-\alpha}$; there exists a finite subset Λ_0 of Λ, for which $\bigwedge_{\lambda \in \Lambda_0} intV_\lambda \notin \mathcal{G}$.
(iii) Every prefilterbase $\mathcal{B}$ on X with $\mathcal{B} \subseteq \mathcal{G}$, S_α-adheres in X.
(iv) Every prefilter $\mathcal{F}$ on X with $\mathcal{F} \subseteq \mathcal{G}$, S_α-adheres in X.

(v) For any prefilter $\mathcal{B}$ on X with $\mathcal{B} \subseteq \mathcal{G}$, $\bigcap_{B\in\mathcal{B}} B^S(\mathcal{G}) \neq \phi$ where for each $B \in \mathcal{B}$, $B^S(\mathcal{G}) = \{x \in X : \text{ for each } U \in \Gamma_\alpha(x), B\, q\, clU\}$.

Proof. $(i) \Rightarrow (ii)$: Let $\{V_\lambda : \lambda \in \Lambda\}$ be a family of fuzzy semi-closed sets such that $\bigwedge_{\lambda\in\Lambda} V_\lambda < K_{1-\alpha}$. Then $1 - \bigwedge_{\lambda\in\Lambda} V_\lambda > K_\alpha$ i.e., $\bigvee_{\lambda\in\Lambda}(1 - V_\lambda) > K_\alpha$. Thus the family $\{(1 - V_\lambda) : \lambda \in \Lambda\}$ is a semi-open α-shading of X. Since X is fuzzy $\mathcal{G}^S_\alpha$-closed, we can find a finite subset Λ_0 of Λ, for which $1 - \bigvee_{\lambda\in\Lambda_0} cl(1-V_\lambda) \notin \mathcal{G}$, so that $\bigwedge_{\lambda\in\Lambda_0} intV_\lambda \notin \mathcal{G}$.

$(ii) \Rightarrow (i)$: Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a semi-open α-shading of X. Then $\{(1-U_\lambda) : \lambda \in \Lambda\}$ is a family of fuzzy semi-closed sets such that $\bigwedge_{\lambda\in\Lambda}(1-U_\lambda) < K_{1-\alpha}$. By assumption of (ii), $\exists$ a finite subset Λ_0 of Λ such that $\bigwedge_{\lambda\in\Lambda_0} int(1 - U_\lambda) \notin \mathcal{G}$. So $1 - \bigvee_{\lambda\in\Lambda_0} cl(U_\lambda) \notin \mathcal{G}$ and consequently X is fuzzy $\mathcal{G}^S_\alpha$-closed.

$(i) \Rightarrow (iii)$: Let $\mathcal{B}$ be a prefilterbase on X with $\mathcal{B} \subseteq \mathcal{G}$. If possible, suppose $\mathcal{B}$ does not S_α-adhere at any crisp point of X. Then for each crisp point $x \in X$, $\exists$ a fuzzy set $U_x \in \Gamma_\alpha(x)$ such that $clU_x\, \bar{q}\, B_x$ for some $B_x \in \mathcal{B}$.
Now the family $\{U_x : x \in X\}$ is a fuzzy semi-open α-shading of X. Since X is fuzzy $\mathcal{G}^S_\alpha$-closed, $\exists$ finitely many points $x_1, x_2, ..., x_n$ in X such that $1 - \bigvee_{i=1}^{n} clU_{x_i} \notin \mathcal{G}$. Again, $clU_x \bar{q} B_x \Rightarrow B_x \leq 1 - clU_x \Rightarrow \bigwedge_{i=1}^{n} B_{x_i} \leq \bigwedge_{i=1}^{n}(1 - clU_{x_i}) \notin \mathcal{G}$. But since $\mathcal{B}$ is a prefilterbase, $\exists$ B in $\mathcal{B}$ such that $B \leq \bigwedge_{i=1}^{n} B_{x_i}$ and then $B \notin \mathcal{G}$ which contradicts the fact that $\mathcal{B} \subseteq \mathcal{G}$.

$(iii) \Rightarrow (ii)$: If possible, suppose $\mathcal{S} = \{S_\lambda : \lambda \in \Lambda\}$ is a collection of fuzzy semi-closed sets in X with $\bigwedge_{\lambda\in\Lambda} S_\lambda < K_{1-\alpha}$ but for each finite subset Λ_0 of Λ, $\bigwedge_{\lambda\in\Lambda_0} intS_\lambda \in \mathcal{G}$. Then clearly $\bigwedge_{\lambda\in\Lambda_0} intS_\lambda \neq 0_X$ (as $0_X \notin \mathcal{G}$), for each finite subset Λ_0 of Λ.
Let us construct $\mathcal{S}_0 = \{\bigwedge_{\lambda\in\Lambda_0} intS_\lambda : \Lambda_0$ is a finite subset of $\Lambda\}$. Then $\mathcal{S}_0$ satisfies the conditions for being a prefilterbase on X. Also $\mathcal{S}_0 \subseteq \mathcal{G}$. Then by (iii), the prefilterbase

$\mathcal{S}_0$, S_α-adheres at some crisp point x_0 (say) in X. That means, for each $U \in \Gamma_\alpha(x_0)$ and each $S \in \mathcal{S}_0$, $clU\,q\,S$. By assumption, $\bigwedge_{\lambda\in\Lambda} S_\lambda < K_{1-\alpha}$. Thus $(1 - \bigwedge_{\lambda\in\Lambda} S_\lambda) > K_\alpha$. So $\bigvee_{\lambda\in\Lambda}(1 - S_\lambda) > K_\alpha$ and then there exists $\beta \in \Lambda$, such that $(1 - S_\beta)(x_0) > \alpha$. Now $intS_\beta \in \mathcal{S}_0$ and $(1 - S_\beta) \in \Gamma_\alpha(x_0)$ implies that $intS_\beta\,q\,cl(1 - S_\beta)$ which is a contradiction.

$(iii) \Rightarrow (iv)$: Follows from the fact that each prefilter is also a prefilterbase.

$(iv) \Rightarrow (iii)$: Let $\mathcal{B}$ be a given prefilterbase with $\mathcal{B} \subseteq \mathcal{G}$. Let $\mathcal{F}$ be the prefilter generated by $\mathcal{B}$. As $\mathcal{B} \subseteq \mathcal{G}$, then $\mathcal{F} \subseteq \mathcal{G}$. By the condition (iv), the prefilter $\mathcal{F}$, S_α-adheres at some point x in X. Then for any $U \in \Gamma_\alpha(x)$ and for each $F \in \mathcal{F}$, $clU\,q\,F$. Now, for each $B \in \mathcal{B}$, $B \in \mathcal{F}$ so that $B\,q\,clU$. Hence the prefilterbase $\mathcal{B}$, S_α-adheres at x in X.

$(i) \Rightarrow (v)$: Let $\mathcal{P}$ be a prefilterbase on X with $\mathcal{P} \subseteq \mathcal{G}$. If possible, let $\bigcap_{P\in\mathcal{P}} P^S(\mathcal{G}) = \phi$, where for each $P \in \mathcal{P}$, $P^S(\mathcal{G}) = \{x \in X : \textit{ for each } U \in \Gamma_\alpha(x), P\,q\,clU\}$. Then for each $x \in X$, there exist $P_x \in \mathcal{P}$ and $U_x \in \Gamma_\alpha(x)$ such that $clU_x\,\bar{q}\,P_x$, i.e., $P_x \leq 1 - clU_x$. Now the family $\{U_x : x \in X\}$ is a fuzzy semi-open α-shading of X. By (i), there exist finitely many points $x_1, x_2, ..., x_n$ in X such that $1 - \bigvee_{i=1}^{n} clU_{x_i} \notin \mathcal{G}$. Then $\bigwedge_{i=1}^{n}(1 - clU_{x_i}) \notin \mathcal{G}$. Thus $\bigwedge_{i=1}^{n} P_{x_i} \leq \bigwedge_{i=1}^{n}(1 - clU_{x_i}) \notin \mathcal{G}$. Since $\mathcal{P}$ is a prefilterbase, $\exists\ P \in \mathcal{P}$ such that $P \leq \bigwedge_{i=1}^{n} P_{x_i} \notin \mathcal{G} \Rightarrow P \notin \mathcal{G}$ contradicting the fact that $\mathcal{P} \subseteq \mathcal{G}$.

$(v) \Rightarrow (iii)$: Let $\mathcal{W}$ be a prefilterbase on X with $\mathcal{W} \subseteq \mathcal{G}$. Then by (v), $\bigcap_{W\in\mathcal{W}} W^S(\mathcal{G}) \neq \phi$, where for each $W \in \mathcal{W}$, $W^S(\mathcal{G}) = \{x \in X : \textit{ for each } U \in \Gamma_\alpha(x), W\,q\,clU\}$. Let $p \in \bigcap_{W\in\mathcal{W}} W^S(\mathcal{G})$. Then $p \in W^S(\mathcal{G})$, for all $W \in \mathcal{W}$. Thus $\forall\, W \in \mathcal{W}$ and for each

$U \in \Gamma_\alpha(p)$, $W\, q\, clU$ which implies that the prefilterbase $\mathcal{W}$, S_α-adheres at the point p in X. □

Finally we want to characterize fuzzy $\mathcal{G}^S_\alpha$-closedness in terms of a fuzzy grill. To prove the main result, we require to prove some necessary results. First we recall the following existing facts:

Result 3.13. *(i)* [3] *For any fuzzy grill $\mathcal{G}$ on an fts (X, τ), $\mathcal{G} \cap \tau$ and $\mathcal{G} \cap \Omega$ are open and closed fuzzy grills on X respectively.*
(ii) [6] *If $\mathcal{F}$ is a fuzzy prefilter on X, then $\mathcal{F} \cap \Omega$ (resp. $\mathcal{F} \cap \tau$) is a closed (resp. open) prefilter on X.*

Definition 3.14. [3] *For any fuzzy grill or a prefilter $\mathcal{G}$, a collection $Sec\,\mathcal{G}$ of fuzzy sets is defined by $Sec\,\mathcal{G} = \{A \in I^X : A\, q\, G \text{ for each G} \in \mathcal{G}\}$.*

Theorem 3.15. [3] *Let (X, τ) be an fts.*
(i) If $\mathcal{G}$ is a fuzzy grill or a prefilter on X, then $A \in Sec\,\mathcal{G} \Leftrightarrow 1 - A \notin \mathcal{G}$.
(ii) If $\mathcal{G}$ is a fuzzy grill (prefilter) on X, then $Sec\,\mathcal{G}$ is a prefilter (resp. fuzzy grill) on X.
(iii) If $\mathcal{G}$ is a closed(resp. an open) fuzzy grill on X, then $Sec\,\mathcal{G} \cap \tau$ (resp. $Sec\,\mathcal{G} \cap \Omega$ is an open (resp. a closed) prefilter on X.
(iv) If $\mathcal{F}$ is an open (resp. closed) prefilter on X, then $Sec\,\mathcal{G} \cap \Omega$ (resp. $Sec\,\mathcal{G} \cap \tau$) is a closed (resp. an open) fuzzy grill on X.

Lemma 3.16. *(i) For an fts (X, τ), every prefilter S_α-adheres in X iff every open prefilter S_α-adheres in X.*
(ii) In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, if every open prefilter $\subseteq \mathcal{G}$, S_α-adheres in X then every prefilter $\subseteq \mathcal{G}$, S_α-adheres in X.

Proof. (i) First let each prefilter S_α-adhere in X. Now since each open prefilter on X forms a base for some prefilter on X, so the necessity part follows.
Conversely let each open prefilter S_α-adhere in X and $\mathcal{F}$ be any prefilter on X. Then $\mathcal{F} \cap \tau$ is an open prefilter on X and by hypothesis, $\mathcal{F} \cap \tau$ also S_α-adheres at some crisp point $p \in X$. Then for each $U \in \Gamma_\alpha(p)$ and for each $F \in \mathcal{F} \cap \tau$, $F\, q\, clU$...........(1)
We claim that for each $U \in \Gamma_\alpha(p)$ and for each $F \in \mathcal{F}$, $F\, q\, clU$.
If not, then there exist some $F_1 \in \mathcal{F}$ and $U \in \Gamma_\alpha(p)$ for which $F_1\, \bar{q}\, clU$, so that $F_1 \leq 1 - clU \in \mathcal{F}$ (as $\mathcal{F}$ is a prefilter). Also $(1 - clU) \in \tau \Rightarrow (1 - clU) \in \mathcal{F} \cap \tau$. Thus for $U \in \Gamma_\alpha(p)$ and $(1 - clU) \in \mathcal{F} \cap \tau$, we have $(1 - clU)\, q\, clU$ [from (1)] which is impossible. Thus $\mathcal{F}$, S_α-adheres in X.

(ii) The proof is similar to that of the converse part of (i) above, where we only note that for any prefilter $\mathcal{F} \subseteq \mathcal{G}$ on X, $\mathcal{F} \cap \mathcal{G}$ is an open prefilter on X such that $\mathcal{F} \cap \mathcal{G} \subseteq \mathcal{G}$. □

Lemma 3.17. *(i) In an fts (X, τ), each fuzzy grill S_α-converges iff every closed fuzzy grill*
S_α-converges in X.
(ii) In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, each fuzzy grill $\subseteq \mathcal{G}$, S_α-converges iff every closed fuzzy grill $\subseteq \mathcal{G}$, S_α-converges in X.

Proof. (i) First let each fuzzy grill S_α-converge in X and $\mathcal{H}$ be any closed fuzzy grill on X. Then by Theorem 3.15(iii), $Sec\,\mathcal{H} \cap \tau$ is an open prefilter on X and hence a prefilterbase for some prefilter on X. Let $\mathcal{F}$ be the prefilter generated by $Sec\,\mathcal{H} \cap \tau$. Then again by Theorem 3.15(ii), $Sec\,\mathcal{F}$ is a fuzzy grill on X and by hypothesis $Sec\,\mathcal{F}$, S_α-converges to some point $p \in X$. Then to each $U \in \Gamma_\alpha(p)$, there corresponds some $S \in Sec\,\mathcal{F}$ such that $S \leq clU$. Thus $clU \in Sec\,\mathcal{F}$ (as $Sec\,\mathcal{F}$ is a fuzzy grill). Hence by definition of $Sec\,\mathcal{F}$, for each $F \in \mathcal{F}$ and for each $U \in \Gamma_\alpha(p)$, we have $clU\, q\, F$.......(1)

We claim that $clU \in \mathcal{H}$, for each $U \in \Gamma_\alpha(p)$. Indeed, if $clU \notin \mathcal{H}$, then $1 - clU \in Sec\,\mathcal{H} \cap \tau$ (by Theorem 3.15(i)]. Thus $1 - clU \in \mathcal{F}$ [as $\mathcal{F}$ is the prefilter generated by $Sec\,\mathcal{H} \cap \tau$]. Therefore by (1), $clU\, q\, (1 - clU)$ which is impossible. So $clU \in \mathcal{H}$, for each $U \in \Gamma_\alpha(p)$ and hence $\mathcal{H}$, S_α-converges in X.

Conversely, let each closed fuzzy grill S_α-converge in X. Let us suppose that $\mathcal{G}$ is any fuzzy grill on X. Then by Result 3.13, $\mathcal{G} \cap \Omega$ is a closed fuzzy grill on X and by assumption, it S_α-converges to some point y in X. Then for each $V \in \Gamma_\alpha(y)$, $\exists$ some $G \in \mathcal{G} \cap \Omega$ such that $G \leq clV$. Thus we can say that for each $V \in \Gamma_\alpha(y)$, $G \leq clV$ for some $G \in \mathcal{G}$ and consequently each fuzzy grill S_α-converges in X.

(ii) The proof is similar to that of (i) and hence left. □

Theorem 3.18. *In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ with $\tau \setminus \{0_X\} \subseteq \mathcal{G}$, X is fuzzy $\mathcal{G}_\alpha^S$-closed iff every fuzzy grill S_α-converges to some point in X.*

Proof. Let $(X, \tau, \mathcal{G})$ be a fuzzy $\mathcal{G}_\alpha^S$-closed space with $\tau \setminus \{0_X\} \subseteq \mathcal{G}$ and $\mathcal{H}$ be any closed fuzzy grill on X. Then by Theorem 3.15(iii), $Sec\,\mathcal{H} \cap \tau$ is an open prefilter on X with $Sec\,\mathcal{H} \cap \tau \subseteq \mathcal{G}$ [as $\tau \setminus \{0_X\} \subseteq \mathcal{G}$]. Then by Theorem 3.12 $(i) \Rightarrow (iii)$, $Sec\,\mathcal{H} \cap \tau$, S_α-adheres at some point $x \in X$ (as each open prefilter forms a base for some prefilter). Thus for each $V \in Sec\,\mathcal{H} \cap \tau$ and for each $U \in \Gamma_\alpha(x)$, $V\, q\, clU$.................(1).
We claim that $clU \in \mathcal{H}$. If not, then $clU \notin \mathcal{H}$ which implies $(1 - clU) \in Sec\,\mathcal{H}$. In fact, $(1 - clU) \notin Sec\,\mathcal{H} \Rightarrow (1 - clU)\, \bar{q} H$ for some $H \in \mathcal{H} \Rightarrow H \leq clU \in \mathcal{H}$ (as $\mathcal{H}$ is a closed fuzzy grill). Now, $(1 - clU) \in \tau$. Thus $(1 - clU) \in Sec\,\mathcal{H} \cap \tau$. Now by (1), we get $clU\, q\, (1 - clU)$ which is absurd. Hence our claim that $clU \in \mathcal{H}$, for all $U \in \Gamma_\alpha(x)$ is established.
Hence $\mathcal{H}$, S_α-converges to some point in X. So by Lemma 3.17, each fuzzy grill

S_α-converges to some point in X.

Conversely, let $\mathcal{F}$ be any open prefilter on X. Then by Theorem 3.15(iv), $Sec\,\mathcal{F}\cap\Omega$ is a closed fuzzy grill on X. Now by the given condition, each fuzzy grill on X, S_α-converges in X. So by Lemma 3.17, every closed fuzzy grill S_α-converges and in particular, $Sec\,\mathcal{F}\cap\Omega$, S_α-converges to some point $y\in X$. Thus for each $U\in\Gamma_\alpha(y)$, $G\le clU$, for some G in $Sec\,\mathcal{F}\cap\Omega$. Then $clU\in Sec\,\mathcal{F}\cap\Omega$ [as $Sec\,\mathcal{F}\cap\Omega$ is a closed fuzzy grill] for each $U\in\Gamma_\alpha(y)$. Again by definition of $Sec\,\mathcal{F}$, $clU\in Sec\,\mathcal{F}\Rightarrow F\,q\,clU$, for each $F\in\mathcal{F}$ and for each $U\in\Gamma_\alpha(y)$. So the open prefilter $\mathcal{F}$, S_α-adheres in X, and by hypothesis, for each open prefilter $\mathcal{F}$, we have $\mathcal{F}\subseteq\mathcal{G}$. Since $\mathcal{F}$ is arbitrary, by Lemma 3.16, every prefilter $\subseteq\mathcal{G}$ on X, S_α-adheres in X and consequently by Theorem 3.12 [$(iv)\Rightarrow(i)$], X is fuzzy $\mathcal{G}^S_\alpha$-closed. □

4. Invariance of fuzzy $\mathcal{G}^S_\alpha$-closedness under fuzzy continuous-like functions

Finally we investigate the invariance property of fuzzy $\mathcal{G}^S_\alpha$-closedness under fuzzy continuous, fuzzy irresolute and fuzzy semi-open functions. To do this we recall the following prerequisites:

Definition 4.1. *A mapping $f : X\to Y$ from an fts X to an fts Y is said to be*
(i) [5] *fuzzy continuous if the inverse image of each fuzzy open set in Y is fuzzy open in X.*
(ii) [2] *fuzzy semi-open if $f(A)$ is fuzzy semi-open in Y, for each fuzzy open set A in X.*
(iii) [18] *fuzzy irresolute if $f^{-1}(B)$ is fuzzy semi-open in X, for any fuzzy semi-open set B in Y.*

Result 4.2. [14] *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following conditions are equivalent:*

(i) f is fuzzy continuous.

(ii) For every fuzzy closed set A in Y, $f^{-1}(A)$ is fuzzy closed in X.

(iii) For any fuzzy set A in X, $f(clA) \leq clf(A)$.

(iv) For any fuzzy set B in Y, $clf^{-1}(B) \leq f^{-1}cl(B)$.

Lemma 4.3. [6] *Let $(X, \tau, \mathcal{G})$ and $(Y, \sigma, \mathcal{H})$ be two fuzzy $\mathcal{G}$-spaces and $f : (X, \tau, \mathcal{G}) \to (Y, \sigma, \mathcal{H})$ be a function. Then*

(i) if f is a surjection, then $f(\mathcal{G}) = \{f(G) : G \in \mathcal{G}\}$ is a fuzzy grill on Y.

(ii) if f is a bijection, then $f^{-1}(\mathcal{H}) = \{f^{-1}(H) : H \in \mathcal{H}\}$ is a fuzzy girll on X.

Definition 4.4. [7] *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy almost open (f.a.o, for short) if $f^{-1}cl(A) \leq clf^{-1}(A)$, for each fuzzy open set A in Y.*

We add one more definition for our purpose.

Definition 4.5. *A map $g : (X, \tau, \mathcal{G}_1) \to (Y, \sigma, \mathcal{G}_2)$ is said to be fuzzy strongly semi-open if for each fuzzy semi-open set S in X, $g(S)$ is fuzzy open in Y.*

Theorem 4.6. *Let $(X, \tau, \mathcal{G}_1)$ and $(Y, \sigma, \mathcal{G}_2)$ be two fuzzy $\mathcal{G}$-spaces and $g : X \to Y$ be a map. If g is fuzzy strongly semi-open, bijective and fuzzy almost open, then X is fuzzy $[g^{-1}(\mathcal{G}_2)]^S_\alpha$-closed whenever Y is fuzzy $[\mathcal{G}_2]^S_\alpha$-closed.*

Proof. Let $\mathcal{S} = \{S_\lambda : \lambda \in \Lambda\}$ be a semi-open α-shading of X. Since g is fuzzy strongly semi-open and onto, $\{g(S_\lambda) : \lambda \in \Lambda\}$ is an open α-shading of Y and so $\{g(S_\lambda) : \lambda \in \Lambda\}$ is also a semi-open α-shading of Y.

Now as Y is fuzzy $[\mathcal{G}_2]^S_\alpha$-closed, there is a finite subset Λ_0 of Λ, for which $1_Y - \bigvee\limits_{\lambda \in \Lambda_0} cl[g(S_\lambda)] \notin \mathcal{G}_2$. Then $g^{-1}[1_Y - \bigvee\limits_{\lambda \in \Lambda_0} cl(g(S_\lambda))] \notin g^{-1}(\mathcal{G}_2)$ i.e., $1_X - g^{-1}(\bigvee\limits_{\lambda \in \Lambda_0} cl[g(S_\lambda)]) \notin g^{-1}(\mathcal{G}_2)$.........(1)

Now, $1_X - g^{-1}(\bigvee_{\lambda\in\Lambda_0} cl[g(S_\lambda)]) = 1_X - \bigvee_{\lambda\in\Lambda_0} g^{-1}[cl(g(S_\lambda))] \geq 1 - \bigvee_{\lambda\in\Lambda_0} cl[g^{-1}g(S_\lambda)]$ (as g is fuzzy almost open) $= 1 - \bigvee_{\lambda\in\Lambda_0} cl(S_\lambda)$ (as g is one-one)
Thus from (1), we get $1 - \bigvee_{\lambda\in\Lambda_0} cl(S_\lambda) \notin g^{-1}[\mathcal{G}_2]$ and hence X is fuzzy $[g^{-1}(\mathcal{G}_2)]^S_\alpha$-closed. □

Theorem 4.7. *Let $(X,\tau,\mathcal{G}_1)$ and $(Y,\sigma,\mathcal{G}_2)$ be two fuzzy $\mathcal{G}$-spaces and $h : X \to Y$ be any fuzzy continuous, fuzzy irresolute and bijective function. Then X is fuzzy $\mathcal{G}^S_\alpha$-closed implies Y is fuzzy $[h(\mathcal{G})]^S_\alpha$-closed.*

Proof. Let $\mathcal{P} = \{P_\beta : \beta \in \Lambda\}$ be any semi-open α-shading of Y. Since h is fuzzy irresolute, then $\{h^{-1}(P_\beta) : \beta \in \Lambda\}$ is a semi-open α-shading of X. By fuzzy $\mathcal{G}^S_\alpha$-closedness of X, we can find a finite subset of Λ, say Λ_0 such that $1_X - \bigvee_{\beta\in\Lambda_0} cl[h^{-1}(P_\beta)] \notin \mathcal{G} \Rightarrow h[1_X - \bigvee_{\beta\in\Lambda_0} cl(h^{-1}(P_\beta))] \notin h(\mathcal{G})$ (h being a bijection) $\Rightarrow 1_Y - \bigvee_{\beta\in\Lambda_0} h[cl(h^{-1}(P_\beta))] \notin h(\mathcal{G}) \Rightarrow 1_Y - \bigvee_{\beta\in\Lambda_0} cl[hh^{-1}(P_\beta)] \notin h(\mathcal{G})$ (by Result 4.2(iii)) $\Rightarrow 1_Y - \bigvee_{\beta\in\Lambda_0} cl(P_\beta) \notin h(\mathcal{G})$. So Y is fuzzy $[h(\mathcal{G})]^S_\alpha$-closed. □

For the next theorem we need the following result:

Theorem 4.8. [18] *A function f from an fts X to an fts Y is fuzzy irresolute if and only if for each fuzzy set A in X, $f(Scl(A)) \leq Scl(f(A))$.*

Theorem 4.9. *Let $f : (X,\tau,\mathcal{G}_1) \to (Y,\sigma,\mathcal{G}_2)$ be a fuzzy irresolute bijection from a fuzzy extremally disconnected fuzzy $\mathcal{G}$-space X to a fuzzy $\mathcal{G}$-space Y. If X is fuzzy $\mathcal{G}^S_\alpha$-closed, then Y is fuzzy $[f(\mathcal{G})]^S_\alpha$-closed.*

Proof. Let $\{V_\lambda : \lambda \in \Lambda\}$ be any semi-open α-shading of Y. Then as f is fuzzy irresolute, $\{f^{-1}(V_\lambda) : \lambda \in \Lambda\}$ is a semi-open α-shading of X. Now by fuzzy $\mathcal{G}^S_\alpha$-closedness of X, we get $1_X - \bigvee_{\lambda\in\Lambda_0} cl[f^{-1}(V_\lambda)] \notin \mathcal{G}$, for some finite subset Λ_0 of Λ.
Also since X is fuzzy extremally disconnected, we can write $1_X - \bigvee_{\lambda\in\Lambda_0} Scl[f^{-1}(V_\lambda)] \notin \mathcal{G}$.....(1) [by Lemma 2.12]. Now $f(1_X - \bigvee_{\lambda\in\Lambda_0} Scl[f^{-1}(V_\lambda)]) \geq 1_Y - f[\bigvee_{\lambda\in\Lambda_0} Scl(f^{-1}(V_\lambda))]$

[by Result 1.8(vi)]= $1_Y - \bigvee_{\lambda\in\Lambda_0} f[Scl(f^{-1}(V_\lambda))] \geq 1_Y - \bigvee_{\lambda\in\Lambda_0} Scl[f(f^{-1}(V_\lambda))]$ (by Theorem 4.8) $\geq 1_Y - \bigvee_{\lambda\in\Lambda_0} Scl(V_\lambda) = 1_Y - \bigvee_{\lambda\in\Lambda_0} cl(V_\lambda)$ [by Lemma 2.12(iii)].
By (1), we get $f(1_X - \bigvee_{\lambda\in\Lambda_0} Scl[f^{-1}(V_\lambda)]) \notin f(\mathcal{G}) \Rightarrow 1_Y - \bigvee_{\lambda\in\Lambda_0} cl(V_\lambda) \notin f(\mathcal{G})$ and hence Y is also fuzzy $[f(\mathcal{G})]^S_\alpha$-closed. □

5. CONCLUSION

The concept S-closedness in a fuzzy topological space was first introduced in [17], which was followed by its further study by many researchers; again Mashhour et al.[13] investigated fuzzy S-closedness in terms of α-shading. Furthermore, in [3] the authors used fuzzy grills for studying certain covering properties in fuzzy setting. The intent of this paper is to blend all these three ideas together and to investigate a new concept of fuzzy S-closedness for a fuzzy $\mathcal{G}$-space, termed fuzzy $\mathcal{G}^S_\alpha$-closedness, where both the concepts of α-shading and fuzzy grill come into play significantly. We do believe in this connection that the investigations of fuzzy $\mathcal{G}^S_\alpha$-closedness can further be generalized to the study of fuzzy $\mathcal{G}^S_\alpha$-closedness of arbitrary fuzzy subsets in a fuzzy $\mathcal{G}$-space.

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