2022

MATHEMATICS — HONOURS

Paper : CC-5

Full Marks : 65

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$ and $\mathbb{R}$ denote the set of natural and real numbers respectively.

Group - A

(Marks : 20)

1. Answer the following multiple choice questions having only one correct option. Choose the correct option and justify : (1+1)×10

(a) Let $l = \lim_{x \to 0+} \sqrt{x-[x]}$ and $m = \lim_{x \to 0-} \sqrt{x-[x]}$. The value of (l, m) is

(i) (0, 1) (ii) (0, 0)

(iii) (1, 0) (iv) (1, 1)

(b) If $f(x) = [x^2]$ on [0, 2.5]; then number of points of discontinuity of $f(x)$ is

(i) one (ii) three

(iii) five (iv) six.

(c) Which one is not uniformly continuous on the indicated intervals?

(i) $f(x) = \begin{cases} x\sin x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ on (0,1) (ii) $f(x) = \tan x$ on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

(iii) $f(x) = \frac{1}{x}$ on $(0,1)$ (iv) $f(x) = \sqrt{x}$ on $(1, \infty)$.

(d) If $f(x) = \begin{cases} x\sin\frac{1}{x^{2023}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ then oscillation of f at 0 is

(i) 1 (ii) 0

(iii) 2023 (iv) infinite.

(e) $\lim_{x \to 0}\left(\frac{1}{x} - \frac{\tan x}{x^2}\right)$ is

(i) $-\frac{1}{2}$ (ii) 0

(iii) $\frac{1}{2}$ (iv) $-\frac{1}{3}$

(f) Which of the following function does not satisfy the Rolle's theorem in $[-1, 1]$?

(i) $|x|$ (ii) $\frac{1}{x^2+4}$

(iii) $\sqrt{x^2+3}$ (iv) x^2.

(g) For a differentiable function f in $[0, 2022]$ with $f(0) = 2022$ and $f(2022) = 0$ which of the following is true?

(i) $f'(c) = 1$ for some $0 < c < 2022$

(ii) $f'(c_1) + f'(c_2) = -2$, for some $0 < c_1 < c_2 < 2022$

(iii) $f'(c_1) + f'(c_2) = 2$, for some $0 < c_1 < c_2 < 2022$

(iv) $f'(c) = 2022$ for some $0 < c < 2022$.

(h) If Cauchy's MVT is applied on two functions $f(x) = x^2$, $g(x) = x$ in $[-1, 1]$, then $\xi \in (-1, 1)$ is equal to

(i) 1 (ii) 0

(iii) -1 (iv) $\frac{1}{2}$.

(i) Let $f(x) = \begin{cases} 2x+3, & x < 0 \\ -3x+1, & x \geq 0 \end{cases}$. Then at $x = 0$, f has

(i) local maximum (ii) local minimum

(iii) neither maximum nor minimum (iv) continuity.

(j) Suppose f be a differentiable function in $\mathbb{R}$ such that f' is monotone. Then which of the following is incorrect?

(i) f' is continuous in $\mathbb{R}$ (ii) f may have a supremum in $\mathbb{R}$

(iii) f may have an infimum in $\mathbb{R}$ (iv) f' may be discontinuous at 0.

Group - B

(Marks : 25)

Answer ***any five*** questions.

2. (a) Let $f: S \to \mathbb{R}$ be a function and 'a' be a limit point of $S\,(\subseteq \mathbb{R})$. If for every sequence $\{x_n\}$ of elements of $S - \{a\}$ converging to 'a', $\{f(x_n)\}$ converges to 'l' $(\in \mathbb{R})$ then prove that $\lim\limits_{x \to a} f(x) = l$.

(b) Evaluate $\lim\limits_{x \to 0} \dfrac{[x^2]}{2x}$, where $[y]$ denotes the largest integer not exceeding y. 3+2

3. (a) Prove that every continuous real valued function over a closed bounded interval is bounded.

(b) Show that, if $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ then $Z(\varphi) = \{x \in \mathbb{R} : \varphi(x) = 0\}$ is closed set in $\mathbb{R}$. 3+2

4. (a) Let $f: A \to \mathbb{R}$, $g: D \to \mathbb{R}$, $c \in A'$ (where A' is the derived set of $A \subseteq \mathbb{R}$) and $f(A) \subset D \subseteq \mathbb{R}$. Show that, if $l \in D$ and g is continuous then $\lim\limits_{x \to c} gf(x) = g(l)$.

(b) Let $D = \left\{\dfrac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\}$ and $f: D \to \mathbb{R}$ where $f\left(\dfrac{1}{n}\right) = n$ and $f(0) = 1$. Find the set of points of discontinuity of f. 3+2

5. If $f: [a, b] \to \mathbb{R}$ be strictly monotonic and continuous on $[a, b]$, then prove that f admits a continuous inverse function. 5

6. (a) Prove or disprove : Any continuous and periodic function on $\mathbb{R}$ is bounded.

(b) Examine, whether the function $f(x) = \sqrt{x}$ satisfies Lipscitz condition in $[0, 1]$. 3+2

7. Prove that, for any monotonically increasing function f on $\mathbb{R}$, if $a < b$ then

$$f(a-) \le f(a+) \le f(b-) \le f(b+).$$ 5

8. Prove that, if f is a continuous function on $\mathbb{R}$, then it has the intermediate value property. Is the converse of this result true? Give reason. 3+2

9. (a) Give an example of real valued functions f and g which are not continuous at a point $c \in \mathbb{R}$ but the product fg is continuous at c.

(b) A function $f: \mathbb{R} \to \mathbb{R}$ satisfies the conditions $f(x + y) = f(x) + f(y)$ for $x, y \in \mathbb{R}$. If f is continuous at one point $c \in \mathbb{R}$, then prove that f is continuous at every point of $\mathbb{R}$. 2+3

Group - C

(Marks : 20)

Answer ***any four*** questions.

10. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 \sin\left(\frac{1}{x}\right), \quad x \neq 0$$
$$= 0, \quad x = 0$$

Show that f is differentiable on $\mathbb{R}$ but f' is not continuous on $\mathbb{R}$. 2+3

11. (a) Let $f(x) = \begin{cases} 1 & \text{for } x \in [0, 1) \\ -1 & \text{for } x \in [1, 2] \end{cases}$. Show that there does not exists a function φ such that $\varphi'(x) = f(x)$.

(b) A function f is twice differentiable on $[a, b]$ and $f(a) = f(b) = 0$. If $f(c) > 0$ for some $c \in (a, b)$, prove that there exists a point ξ in (a, b) such that $f'(\xi) = 0$. 2+3

12. Let $f: (a, b) \to \mathbb{R}$, $c \in (a, b)$ and $f'(c)$ exists.

(a) Show that if $f(c) \neq 0$, then $\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{(f(c))^2}$.

(b) Prove that $\underset{h \to 0}{\text{Lt}} \frac{f(c+h) - f(c-h)}{2h} = f'(c)$. 2+3

13. (a) Show that $\varphi(x) = \frac{\tan x}{x}$ is a strictly increasing function on $\left(0, \frac{\pi}{2}\right)$.

(b) Using Lagrange's mean value theorem prove that $1 + x < e^x < 1 + xe^x \;\; \forall \; x > 0$. 2+3

14. (a) If f'' is continuous in the interval $[a, a+h]$ and $f''(a) \neq 0$, prove that $\underset{h \to 0}{Lt}\, \theta = \frac{1}{2}$, where θ is given by $f(a+h) = f(a) + hf'(a + \theta h)$, $0 < \theta < 1$.

(b) Find $\lim_{x \to 0+} x^x$. 3+2

15. A conical tent of given capacity (volume) has to be constructed. Find the ratio of the height to the radius of the base for the minimum amount of canvas required for the tent. 5

16. Find the Taylor's infinite series expansion of $\sin x$, about $x = 0$ together with the range of validity. 5