2023

MATHEMATICS — HONOURS

Paper : CC-3

(Real Analysis)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ denote the set of all natural, rational and real numbers respectively.

Notations and symbols have their usual meanings.

1. Answer all the following multiple choice questions. For each question 1 mark is for choosing the correct option and 1 mark is for justification. (1+1)×10

(a) The derived set of the set $S=\left\{\frac{n-1}{n+1}\,\middle|\, n=1,2,...\right\}\cup\{2,3\}$ is

(i) $\{1\}$ (ii) $\{1\}\cup[2,3]$

(iii) $\{0\}$ (iv) $\{2, 3\}$.

(b) The set $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{n}{n+1}\right)$ is

(i) open (ii) closed

(iii) both open and closed (iv) neither open nor closed.

(c) A countable set of irrationals which is dense in $\mathbb{R}$ is

(i) the set $\mathbb{R}\setminus\mathbb{Q}$ of all irrationals (ii) $\mathbb{Q}\cup\{\sqrt{p} : p$ is a prime$\}$

(iii) $\{\sqrt{p} : p$ is a prime$\}$ (iv) $\{\sqrt{2}-r : r\in\mathbb{Q}\}$.

(d) Let $A=[0,1]\cap\mathbb{Q}$ and $B=\left\{\frac{1}{n}:n\in\mathbb{N}\right\}$. Then the set $A-B=\{x-y : x\in A,\ y\in B\}$ is

(i) empty (ii) finite

(iii) enumerable (iv) uncountable.

(e) The sequence $\left\{\left(\frac{2}{3}\right)^n+\left(\frac{3}{2}\right)^n\right\}$

(i) converges to 0
(ii) converges to 1
(iii) converges to $\frac{13}{6}$
(iv) diverges to $+\infty$.

(f) Let $u_n=\sin\frac{n\pi}{2}, n\geq 1$. Then the subsequence $\{u_{2n-1}\}$

(i) is a convergent subsequence
(ii) diverges to $+\infty$
(iii) is a convergent subsequence and converges to 1
(iv) is oscillatory.

(g) Let $u_n=\cos\frac{n\pi}{2}, v_n=\sin\frac{n\pi}{2}$. Then, $\lim\limits_{n\to\infty}\sup(u_n+v_n)$ is equal to

(i) 0
(ii) 1
(iii) 2
(iv) -1.

(h) Which of the following is not a Cauchy sequence?

(i) $\left\{\frac{(-1)^n}{n}\right\}$
(ii) $\left\{\left(1+\frac{1}{n}\right)^n\right\}$
(iii) $\left\{n^{\frac{1}{n}}\right\}$
(iv) $\left\{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right\}$.

(i) The series $1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots$ converges for

(i) $p>1$
(ii) $p<1$
(iii) $p\leq 1$
(iv) $p\geq 1$.

(j) The infinite series $\frac{\sin\frac{\pi}{2}}{1\cdot 2}-\frac{\sin\frac{\pi}{2^2}}{2\cdot 3}+\ldots+(-1)^{n+1}\frac{\sin\frac{\pi}{2^n}}{n\cdot(n+1)}+\ldots$ is

(i) divergent
(ii) oscillatory
(iii) conditionally convergent
(iv) absolutely convergent.

Unit - 1

Answer ***any four*** questions.

2. State LUB Axiom of the set of real numbers, $\mathbb{R}$. Hence deduce the Archimedean property of $\mathbb{R}$. 1+4

3. (a) Prove or disprove : A countable set cannot have uncountable number of limit points.

(b) Show that the set $\{x \in \mathbb{R} : \sin x = 0\}$ is countable. 3+2

4. State and prove Bolzano Weierstrass Theorem on limit points. 5

5. (a) Find all the isolated points of $S = \left\{ \frac{n-1}{n+1} \Big/ n = 1,2,3,... \right\} \cup (2,3)$.

(b) Prove or Disprove :

$$S = \bigcup_{n=1}^{\infty} I_n, \text{ where } I_n = \left\{ x \in \mathbb{R} : \left(\frac{1}{3}\right)^n \le x \le 1 \right\}; \text{ is a closed set.}$$ 2+3

6. (a) Give an example of an unbounded countable subset of $\mathbb{R}$ having no limit points.

(b) Show that the set S is an open set where $S = \{x \in \mathbb{R} : |x-1| + |x-2| < 3\}$. 2+3

7. Prove or Disprove : Every infinite bounded set of rational numbers has a limit point in $\mathbb{Q}$. 5

8. (a) Consider the intervals $S = (0, 2]$ and $T = [1, 3)$. Let S^o and T^o be the set of interior points of S and T respectively. Then find the set of all interior points of $S \setminus T$.

(b) Prove or disprove : Every uncountable set has a limit point. 2+3

Unit - 2

Answer ***any four*** questions.

9. (a) Prove that if the sequence $\{x_n\}_n$ converges to l, then the sequence $\{|x_n|\}_n$ converges to $|l|$. Is the converse true? Justify your answer.

(b) Give examples of two non-convergent sequence $\{x_n\}_n$ and $\{y_n\}_n$ such that sequences $\{x_n + y_n\}$ and $\{x_n y_n\}$ both converge. (2+1)+(1+1)

10. (a) Prove or disprove : If $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers such that $\lim_{n\to\infty} x_n = 0$ and $\{y_n\}$ is a bounded sequence, then $\lim_{n\to\infty} (x_n y_n) = 0$.

(b) If $a > 0$, prove that $\left\{ a^{\frac{1}{n}} - 1 \right\}$ is a null sequence. 3+2

Please Turn Over

11. (a) Define Cauchy sequence of real numbers. Using the definition show that $\left\{\frac{n}{n+1}\right\}$ is a Cauchy sequence.

(b) Show that every Cauchy sequence is bounded. (1+2)+2

12. Let $\{[a_n, b_n]\}$ be a sequence of closed and bounded intervals, such that

(a) $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$ and (b) $\lim_{n\to\infty}(b_n - a_n) = 0$.

Show that $\bigcap_{n=1}^{\infty}[a_n, b_n]$ contains exactly one element. 5

13. Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$ for all $n \geq 1$ is convergent. Find the limit of the sequence $\{x_n\}$. 3+2

14. Prove that every sequence of real numbers has a monotonic subsequence. 5

15. (a) State the Sandwich theorem.

(b) Prove that $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+...+\frac{1}{\sqrt{n^2+n}}\right)=1$. 2+3

Unit - 3

Answer *any one* question.

16. (a) Examine the convergence of the series $\sum_{n=1}^{\infty}\frac{\cos(3^n\pi)}{2^n}$.

(b) Test the convergence of the series $\frac{1}{2}+2+\frac{1}{2^2}+2^2+\frac{1}{2^3}+2^3+....$ 3+2

17. (a) By comparison test, show that the series $\frac{1}{1\cdot 2^2}+\frac{1}{2\cdot 3^2}+\frac{1}{3\cdot 4^2}+...$ is a convergent series.

(b) Show that the series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...$ is absolutely convergent. 2+3
