2023

MATHEMATICS — HONOURS

Paper : CC-4

(Group Theory - I)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meanings.

1. Answer all the multiple choice questions each having only one correct answer. Each question carries 2 marks, 1 mark for correct answer and 1 mark for justification. (1+1)×10

(a) The order of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}$ is

(i) 4 (ii) 8

(iii) 6 (iv) 2.

(b) Let $G = \langle a \rangle$ be a cyclic group of order 20. Then

(i) $\langle a^7 \rangle = \langle a^{14} \rangle$ (ii) $\langle a^7 \rangle = G$

(iii) $\langle a^7 \rangle \neq G$ (iv) $\langle a^7 \rangle = \langle a^6 \rangle$.

(c) Let G be a non-abelian group of order 10. The number of elements of order 5 in the group is

(i) 1 (ii) 2

(iii) 4 (iv) 9.

(d) Which of the following statements is false?

(i) $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group

(ii) $\mathbb{Z}/56\mathbb{Z}$ has 24 generators

(iii) Order of the element $(7 + 56\mathbb{Z})$ in the group $\mathbb{Z}/56\mathbb{Z}$ is 8

(iv) The group $\mathbb{Z}/56\mathbb{Z}$ has 10 subgroups.

(e) The maximum order of an element in S_{11} is

(i) 18 (ii) 28

(iii) 30 (iv) 60.

(f) The order of alternating group of degree n is

(i) $\frac{n!}{1!}$ (ii) $\frac{n!}{2!}$

(iii) $\frac{n!}{3!}$ (iv) $\frac{n!}{4!}$.

(g) Which of the following statements is true for the semi-group $(G, \cdot)$, where

$$G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R} \text{ and } a \neq 0 \right\}?$$

(i) G does not contain identity element

(ii) $(G, \cdot)$ is a group

(iii) $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ is the identity element of G.

(iv) $(G, \cdot)$ is not a group.

(h) The order of the quotient group $4\mathbb{Z} / 12\mathbb{Z}$ is

(i) 1 (ii) 3

(iii) 4 (iv) 12.

(i) The identity element of the group $(\mathbb{Z}, \circ)$, where $a \circ b = a + b + 1$, $a, b \in \mathbb{Z}$ is

(i) 0 (ii) 1

(iii) – 1 (iv) 2.

(j) Which of the following pair of groups is isomorphic?

(i) $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ (ii) $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$

(iii) $(\mathbb{R}^*, \cdot)$ and $(\mathbb{R}, +)$ (iv) $(\mathbb{Q}, +)$ and $(\mathbb{R}^*, \cdot)$,

where $[\mathbb{R}^* = \mathbb{R} - \{0\}]$.

Unit - I

2. Answer ***any two*** questions :

(a) Let G be a group and H be a non-empty subset of G. A relation ρ is defined on G by "$a \rho b$ if and only if $a^{-1} b \in H$". Prove that H is a subgroup of G if and only if ρ is an equivalence relation on G. 5

(b) (i) Let G be group of even order. Show that there exists $a \in G$, such that $a \neq e$ and $a^2 = e$.

(ii) If a, b be two elements in a group G such that $a^4 = e$ and $a^2b = ba$, then show that $a = e$. 3+2

(c) Prove that intersection of two subgroups of a group forms a subgroup of that group. Show by an example that union of two subgroups of a group is not necessarily a subgroup of that group. 3+2

(d) Let $(G, *)$ be a group and H, K be two subgroups of G. Prove that HK is a subgroup of G if and only if $HK = KH$. 5

Unit - II

3. Answer ***any four*** questions :

(a) (i) Find the order of the permutation $(1\ 2\ 3\ 4) \circ (4\ 5\ 6)$ in S_7.

(ii) Find the number of distinct cycles of length 3 in S_6. 2+3

(b) Define cyclic group. If an abelian group G of order 10 contains an element of order 5, prove that G must be a cyclic group. 1+4

(c) State and prove Lagrange's theorem. 2+3

(d) Using results of group theory, prove that for every positive integer n, $\sum_{d|n} \phi(d) = n$, the sum being taken for all positive divisors d of n. 5

(e) Let $(G, *)$ be a finite group of order n. Prove that G is cyclic if and only if there exists an element a in G such that $O(a) = n$. 5

(f) (i) Express the permutation $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 3 & 1 & 7 & 6 & 5 & 8 \end{pmatrix}$ as a product of disjoint cycles and then find the order of ρ.

(ii) If H is the smallest subgroup of the group $(\mathbb{Z}, +)$ such that $4, 6 \in H$, then prove that $H = 2\mathbb{Z}$. (2+1)+2

(g) (i) Show that S_n is non-commutative for all $n \geq 3$.

(ii) Let n be an odd positive integer and $\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix} \in S_n$.

Show that $(a_1 - 1)(a_2 - 2).....(a_n - n)$ is an even number. 2+3

Unit - III

4. Answer *any three* questions :

(a) State and prove first isomorphism theorem. 5

(b) (i) Let $(G, \circ)$ and $(G', *)$ are two groups and $\phi : G \to G'$ is an epimorphism. If $(G, \circ)$ is cyclic, then prove that $(G', *)$ is cyclic.

(ii) Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \text{ and } ac \neq 0 \right\}$ be a group under matrix multiplication. Then prove that $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$ is a normal subgroup of G. 2+3

(c) (i) Prove that there does not exist an onto homomorphism from the group $(\mathbb{Z}_6, +)$ to the group $(\mathbb{Z}_4, +)$.

(ii) Prove that a non-commutative group of order 10 has a trivial centre. 2+3

(d) State Cayley's theorem. Consider the group $G = \{1, -1, i, -i\}$ with respect to usual multiplication of complex numbers. Find a subgroup H of S_4 such that $G \cong H$. 1+4

(e) (i) Let G be a group and $f : G \to G$ be defined by $f(a) = a^n$ for all $a \in G$, where n is a positive integer. Suppose f is an isomorphism. Prove that $a^{n-1} \in \mathbb{Z}(G)$ for all $a \in G$.

(ii) Let G be a group and $f : G \to G$ be defined by $f(a) = a^3$ for all $a \in G$, be an isomorphism. Prove that G is commutative. 3+2
