2022

MATHEMATICS — HONOURS

Paper : CC-12

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Choose the correct answer with proper justification (1 mark for right answer and 1 mark for justification) : 2×10

(a) Which of the following may be order of an element of the group $S_3 \times S_3$?

(i) 4 (ii) 6 (iii) 9 (iv) 18

(b) Which of the following is the possible number of Abelian groups of order 12?

(i) 1 (ii) 2 (iii) 3 (iv) 4

(c) If $(\mathbb{Z}, +)$ is the additive group of all integers, then which of the following is the possible order of Aut $\mathbb{Z}$?

(i) Infinite (ii) 2 (iii) 1 (iv) Greater than 2

(d) Which of the following is the order of any non-identity element of $\mathbb{Z}_3 \times \mathbb{Z}_3$?

(i) 3 (ii) 6 (iii) 9 (iv) 2

(e) If Z_2 and Z_3 be two groups under addition modulo 2 and 3 respectively, then which of the following is true?

(i) $Z_2 \times Z_2 \cong Z_4$ (ii) $Z_2 \times Z_3 \cong Z_6$

(iii) Both (i) and (ii) are true (iv) None of the above is true

(f) If $V(F)$ is an inner product space and $V^{\perp}$ is the orthogonal complement of V, then

(i) $V^{\perp} = \phi$ (ii) $V^{\perp} = \{\theta\}$ (iii) $V^{\perp} = V$ (iv) $V \cap V^{\perp} = \phi$

(g) If a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x, y, z) = (x + y - z, x - z, y - z)$; $\forall (x, y, z) \in \mathbb{R}^3$, then $T^*(x, y, z) =$

(i) $(x + y, x + z, -x - y - z)$ (ii) $(x + y, x + z, y + z)$

(iii) $(x + y, x + z, -x - y + z)$ (iv) $(x + y, y + z, -x - y - z)$

(h) Which of the following is the signature of the quadratic form $xy + yz + zx$?

(i) 1 (ii) -1 (iii) 2 (iv) -2

(i) Which of the following is the dimension of the orthogonal complement of the row space of the matrix A given by $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{pmatrix}$?

(i) 1 (ii) 2 (iii) 3 (iv) 4

(j) The minimal polynomial of the zero linear operator on an n-dimensional vector space is

(i) x^n (ii) x^{n-1} (iii) x (iv) none of these

Unit – I

(Group Theory)

2. Answer ***any four*** questions :

(a) (i) Let G_1 and G_2 be two groups. Prove that $G_1 \times G_2$ is commutative if and only if both G_1 and G_2 are commutative.

(ii) Prove or disprove : Every group of order 2022 is commutative. 3+2

(b) (i) For a group G, prove that $Inn(G)$ is a normal subgroup of $Aut(G)$.

(ii) Prove or disprove : If G is a cyclic group, then $Aut(G)$ is also a cyclic group. 3+2

(c) Let $f : G \to G$ be a homomorphism. If f commutes with every inner automorphism of G, then prove that

(i) $K = \left\{x \in G; f^2(x) = f(x)\right\}$ is a normal subgroup of G.

(ii) G/K is abelian. 3+2

(d) Prove that $Inn(S_3) = Aut(S_3)$. 5

(e) (i) Let G be a group. Show that the mapping $f : G \to G$ defined by $f(a) = a^{-1}$ for all $a \in G$ is an automorphism if and only if G is an abelian group.

(ii) Show that there exist groups G and H such that $Aut(G) \simeq Aut(H)$ though $G \neq H$. 3+2

(f) (i) Prove that there is no finite group G such that $Aut(G) \cong Z_p$ where p is an odd prime.

(ii) Let G be a group such that $Z(G) = \{e\}$. Prove that $Z(Aut\ G) = \{id\}$. 3+2

(g) (i) Show that any abelian group of order 105 contains a cyclic subgroup of order 15.

(ii) Prove that every non-cyclic group of order p^2, p is a prime number, is isomorphic to external direct product of two cyclic groups each of order p. 3+2

Unit – II

(Linear Algebra)

3. Answer *any five* questions :

(a) (i) If $V(F)$ is an inner product space and A, B are two subsets of V such that $A \subset B$, then prove that $B^{\perp} \subset A^{\perp}$ where $A^{\perp}$ and $B^{\perp}$ are orthogonal complements of A and B respectively.

(ii) If $\{\beta_1, \beta_2, \ldots\ldots\ldots, \beta_r\}$ be an orthogonal set of vectors in an inner product space $V(\mathbb{R})$, then prove that for any vector α in V, $\|\alpha\|^2 \geq c_1^2 + c_2^2 + \ldots\ldots\ldots + c_r^2$ where c_i is the scalar component of α along β_i, $i = 1, 2, \ldots \ldots., r$. 2+3

(b) Let P_3 be the inner product space of all real polynomials of degree $\leqslant 3$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt;\ f, g \in P_3$ and also let W be the subspace of P_3 with basis $\{1, t^2\}$. Find a basis for $W^{\perp}$. 5

(c) Using Gram Schimdt orthonormalisation process, find an orthonormal basis corresponding to the basis $\{(1, 0, 1), (1, 0, -1), (0, 3, 4)\}$ in $\mathbb{R}^3(\mathbb{R})$ using standard inner product. 5

(d) (i) Find the Hessian matrix of the function $f(x, y) = x^3 - 2xy - y^6$ at (1, 2).

(ii) Let V be an n-dimensional vector space over the field F. Find the minimal polynomial of the identity operator $I_V : V \to V$. 2+3

(e) Let W be a subspace of $\mathbb{R}^4$ spanned by $(1, 2, -3, 4)$, $(1, 3, -2, 6)$ and $(1, 4, -1, 8)$. Find a basis of the annihilator of W. 5

(f) Find the Jordan normal form of $\begin{pmatrix} 4 & -1 & 1 \\ 4 & 0 & 2 \\ 2 & -1 & 3 \end{pmatrix}$ over the field of reals. 5

(g) Diagonalise the matrix $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$. 5

(h) Reduce the equation $9x^2 - 24xy + 16y^2 + 2x - 11y + 16 = 0$ to its canonical form and determine the nature of the conic. 5