2021

MATHEMATICS — HONOURS

Paper : CC-13

(Metric Space and Complex Analysis)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[$\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$ denote the set of all natural, real, rational and complex numbers respectively.]

(Notations and symbols have their usual meanings).

1. Answer ***all*** the following multiple choice questions. For each question **1** mark for choosing correct option and **1** mark for justification. 2×10

(a) Let (X, d) be a metric space and $A, B \subseteq X$. Choose the statement which is not true.

(i) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$ (ii) $A^{o} \cup B^{o} \subseteq (A \cup B)^{o}$

(iii) $\partial(A \cap B) = \partial A \cap \partial B$ (iv) $d(A, B) = d(\overline{A}, \overline{B})$.

[∂A denotes boundary of A, $d(A, B)$ denotes the distance between A, B.]

(b) Two metrices d and d^* are defined on $\mathbb{R}^2$ as follows :

$$\text{For all } \hat{x} = (x_1, y_1), \hat{y} = (x_2, y_2) \in \mathbb{R}^2$$
$$d(\hat{x}, \hat{y}) = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 \right]^{1/2}$$
$$d^*(\hat{x}, \hat{y}) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Then which of the following is true?

(i) $d = d^*$ (ii) d and d^* are equivalent

(iii) d and d^* are not equivalent (iv) $(\mathbb{R}^2, d^*)$ is a submetric space of $(\mathbb{R}^2, d)$.

(c) Let $A = \{(x, y) : x,\ y \in \mathbb{R},\ x \notin \mathbb{Q}\ \ or\ \ y \notin \mathbb{Q}\}$. Then which of the following is true with usual metric on $\mathbb{R}^2$?

(i) A is open but not compact in $\mathbb{R}^2$ (ii) A is not open but compact in $\mathbb{R}^2$

(iii) A is neither open nor compact in $\mathbb{R}^2$ (iv) A is both open and compact in $\mathbb{R}^2$.

(d) Consider $\mathbb{R}^2$ with usual metric.

Let $A = \left\{\left(\frac{1}{n}, 0\right) \in \mathbb{R}^2 : n \in \mathbb{N}\right\} \cup \{(0,0)\}$ and

$B = \left\{\left(x, \sin\frac{1}{x}\right) \in \mathbb{R}^2 : 0 < x < 1\right\}$. Then which of the following is correct?

(i) Both A, B are connected
(ii) A is connected, B is disconnected
(iii) B is connected, A is disconnected
(iv) Both A, B are disconnected.

(e) Let A be a subset of a metric space (X, d). Then $\{x \in X : d(x, A) = 0\}$ is :

(i) equal to $\bar{A}$ and is compact
(ii) equal to $\bar{A}$ but not necessarily compact
(iii) equal to A
(iv) equal to $\{0\}$.

(f) Let $T(z) = \dfrac{az+b}{cz+d}$ be a bilinear transformation. Then ∞ is a fixed point of T if and only if

(i) $a = 0$
(ii) $b = 0$
(iii) $c = 0$
(iv) $d = 0$.

(g) Let $f(z) = |z|^2;\ z \in \mathbb{C}$. Then f is

(i) continuous everywhere but differentiable nowhere
(ii) differentiable only at $z = 0$
(iii) continuous nowhere
(iv) differentiable everywhere.

(h) The radius of convergence of the power series $\sum (4+3i)^n z^n$ is

(i) 5
(ii) $\frac{1}{5}$
(iii) 4
(iv) $\frac{1}{4}$.

(i) Value of the integral $\int_C \sec(z)\, dz$, where C is the unit circle with centre at origin, is :

(i) 2
(ii) 0
(iii) 1
(iv) –5.

(j) Let $I = \int_{\gamma} z^2 dz$, where γ is along the real-axis from 0 to 1 and then along the line parallel to the imaginary-axis from 1 to $1 + 2i$. Which of the followings is true?

(i) $I = -\dfrac{11+2i}{3}$ (ii) $I = \dfrac{11-2i}{3}$

(iii) $I = \dfrac{-11+2i}{3}$ (iv) $I = \dfrac{11+2i}{3}$.

Unit - 1

(Metric Space)

Answer ***any five*** questions.

2. Let $\mathbb{R}_\infty$ be the extended set of real numbers. The function d defined by $d(x, y) = |f(x) - f(y)|, \forall x, y \in \mathbb{R}_\infty$, where $f(x)$ is given by

$$f(x) = \begin{cases} \dfrac{x}{1+|x|}, & when \quad -\infty < x < \infty \\ 1, & when \quad x = \infty \\ -1, & when \quad x = -\infty \end{cases}$$

Show that $(\mathbb{R}_\infty, d)$ is a bounded metric space. 5

3. Let $\{a_n\}$ and $\{b_n\}$ be sequences in a metric space (X, d). Write

$$x_n = d(a_n, b_n) \,\forall\, n \in \mathbb{N}$$

If $\{a_n\}$ is a Cauchy sequence and $x_n \to 0$ with respect to usual metric on $\mathbb{R}$, then prove that $\{b_n\}$ is a Cauchy sequence. Is this true if $\{x_n\}$ converges to nonzero limit? Justify. 3+2

4. Let (X, d) be a complete metric space and $\{F_n\}$ be a decreasing sequence of nonempty closed subsets of X such that diam $(F_n) \to 0$ as $n \to \infty$. Then show that the intersection $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point. If diam $(F_n) \nrightarrow 0$ as $n \to \infty$, what would happen? Justify your answer. 3+2

5. Prove or disprove : Let (X, d) be a metric space and A be a closed and bounded subset of X. Then A is compact. 5

6. (a) Prove that a metric space (X, d) having the property that every continuous map $f : X \to X$ has a fixed point, is connected.

(b) Let (X, d) be a complete metric space and $T : X \to X$ be a contraction on X. Then for $x \in X$, show that the sequence $\{T^n x\}$ is a convergent sequence. 2+3

7. (a) Prove that the space $\mathbb{Q}$ of rational numbers with subspace metric of the usual metric of $\mathbb{R}$ is not connected.

(b) Prove that $(a, b]$ is connected with usual metric of $\mathbb{R}$. 2+3

8. Let (X, d_1) and (Y, d_2) be two metric spaces and $f : (X, d_1) \to (Y, d_2)$ be uniformly continuous. Show that if $\{x_n\}$ is a Cauchy sequence in (X, d_1) then so is $\{f(x_n)\}$ in (Y, d_2). Is it true if f is only continuous? Justify. 3+2

9. Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction mapping with Lipschitz constant t $(0 < t < 1)$. If $x_0 \in X$ is unique fixed point of f, show that

$$d(x, x_0) \le \frac{1}{1-t} d\big(x, f(x)\big), \ \forall x \in X \qquad 5$$

Unit - 2

(Complex Analysis)

Answer ***any four*** questions.

10. (a) Let z_1 and z_2 be the images in the complex plane of two diametrically opposite points on the Riemann sphere under stereographic projection. Then show that

$$z_1 \overline{z_2} = -1 .$$

(b) Prove or disprove : The image of the circle $|z| = r$ $(r \neq 1)$ under the transformation $w = z + \frac{1}{z}$ is an ellipse. 2+3

11. Let $f :\ \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \frac{(\overline{z})^2}{z}$ for $z \neq 0$ and $f(0) = 0$. Show that the Cauchy Riemann equations are satisfied at $z = 0$, but the derivative of f fails to exist there. 5

12. (a) If f is analytic function of $z = x + iy \in \mathbb{C}$ and $\overline{z} = x - iy$, then show that $\frac{\partial f}{\partial \overline{z}} = 0$.

(b) Let f be analytic on a region G. If f assumes only real values on G, then show that f is a constant function. 3+2

13. Let f be an analytic function in a region G. Show that $Re(f)$ satisfies the following equation :

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0.$$ 5

14. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Show that f is differentiable on $|Z| < R$. Show that $\frac{df}{dz}=\sum_{n=1}^{\infty} na_n z^{n-1}$ and it has radius of convergence R. 5

15. (a) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n = \frac{e^{in\pi}}{n}$.

(b) Let $f(z)=\bar{z}$ and γ is the semicircle from 1 to –1 passing through i. Evaluate $\int_\gamma f(z)dz$. 2+3

16. (a) Evaluate : $\int_{|z|=2}\frac{e^z+z^2}{z-1}dz$

(b) Show that $\left|\int_\gamma \frac{dz}{z^2+4}\right| \le \frac{\pi R}{(R^2-4)}$, where $\gamma(t)=\text{Re}^{it}$ for $0 \le t \le \pi$ and $R > 2$. 3+2