2021

MATHEMATICS — HONOURS

Paper : DSE-B(2)-1

(Point Set Topology)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer all the following multiple choice questions. For each question, **1** mark for choosing correct option and **1** mark for justification. 2×10

(a) If τ_1 and τ_2 are the topologies on $\mathbb{R}^2$ generated by the base β_1 of interiors of all circular regions in $\mathbb{R}^2$ and the base β_2 of interiors of all rectangular regions in $\mathbb{R}^2$ respectively, then

(i) τ_1 is a proper subset of τ_2 (ii) τ_2 is a proper subset of τ_1

(iii) $\tau_1 = \tau_2$ (iv) $\tau_1 \cap \tau_2 = \{\mathbb{R}^2, \emptyset\}$.

(b) Let (X, τ) be a topological space and A be a non-empty subset of X such that every non-empty open subset of X intersects A. Then which of the following is true?

(i) A must be equal to X (ii) A is dense in X

(iii) $A = \overline{A}$ (iv) A must be an open set.

(c) Let (X, τ) be a topological space and A be a non-empty proper subset of X such that the boundary of A is an empty set. Then which of the following is false?

(i) A contains all of its limit points

(ii) Every point of A is an interior point

(iii) The boundary of $(X \setminus A)$ is an empty set

(iv) A is closed but may not be an open set.

(d) An uncountable set with cofinite topology is

(i) both T_1 and first countable space.

(ii) both T_2 and first countable space.

(iii) a first countable space but not a T_2 space.

(iv) neither first countable nor a T_2 space.

(e) Let $f:(\mathbb{R},\tau_u)\to(\mathbb{R},\tau_u)$ be a continuous map (where τ_u denotes the usual topology on $\mathbb{R}$) and $Z(f)=\{x\in\mathbb{R}: f(x)=0\}$. Which of the following is true?

(i) $Z(f)$ must be a closed set
(ii) $Z(f)$ must be compact
(iii) $Z(f)$ must be an open set
(iv) $Z(f)$ must be connected.

(f) The number of T_1 topologies that can be defined on a finite set with n elements is

(i) 1
(ii) n
(iii) 2^n
(iv) $n-1$.

(g) Which of the following statements is not correct for the discrete topology τ_d on $\mathbb{R}$?

(i) τ_d is the largest topology on $\mathbb{R}$
(ii) $(\mathbb{R}, \tau_d)$ is compact
(iii) $(\mathbb{R}, \tau_d)$ is first countable
(iv) For every subset A of $\mathbb{R}$, $A^{o}=\overline{A}$,
where A^{o} and $\overline{A}$ denotes the interior and closure of A in $(\mathbb{R}, \tau_d)$.

(h) If $\tau=\{\phi,\{a\},\{a,b\},X\}$ is a topology on $X=\{a,b,c\}$, then (X,τ) is

(i) compact and Hausdorff
(ii) compact but not Hausdorff
(iii) only Hausdorff
(iv) neither compact nor Hausdorff.

(i) Which of the following statements is not true?

(i) $\mathbb{R}$ with usual topology is homeomorphic with the subspace topology on $(-1, 1)$.
(ii) $\left[-1,\frac{1}{2}\right)$ is open in $[-1, 1]$ with respect to the subspace topology from the usual topology on $\mathbb{R}$.
(iii) $[-1, 1]$ is homeomorphic with $\{(x,y)\in\mathbb{R}^2: x^2+y^2=1\}$, where both the sets are endowed with the subspace topology from the usual topology on $\mathbb{R}$ and product topology on $\mathbb{R}^2$ respectively.
(iv) $[-1, 1]$ is homeomorphic with $[0, 1]$, where both the sets are endowed with the subspace topology from the usual topology on $\mathbb{R}$.

(j) Let $X=\mathbb{N}\times\mathbb{Q}$ with the subspace topology of $\mathbb{R}^2$ and $P=\left\{\left(n,\frac{1}{n}\right): n\in\mathbb{N}\right\}$. Which of the following statements is true?

(i) P is closed but not open
(ii) P is open but not closed
(iii) P is both open and closed
(iv) P is neither open nor closed.

Unit - 1

(Marks : 20)

Answer ***any four*** questions.

2. Let (X, τ) be the topological product of the family of topological spaces $\{(X_i, \tau_i) : i = 1, 2, ..., n\}$ and $p_i : X \to X_i$ denote the i th projection map $\forall i = 1, 2, ..., n$. Prove that

(a) p_i is an open map for each i

(b) τ is the smallest topology on X such that each p_i is continuous. 2+3

3. Prove that a topological invariant is a metric invariant. Is the converse true? Justify. 3+2

4. Let (X, d) be a metric space and A be a nonempty subset of X. Prove that the function $f_A : (X, \tau(d)) \to \mathbb{R}$ defined by $f_A(x) = \inf \{d(x, a) : a \in A\}$, $\forall x \in X$ is continuous on X (where $\tau(d)$ denotes the metric topology on X induced by d). Hence prove that for any $A \subseteq X$,

$$\bar{A} = \{x \in X : d(x, A) = 0\} \text{ in } (X, \tau(d))$$ 3+2

5. (a) τ is the usual topology on $\mathbb{R}$ and $\tau' = \{A \cup B : A \in \tau, B \subseteq \mathbb{R} \setminus \mathbb{Q}\}$. Prove that τ' is a topology on $\mathbb{R}$ which is finer than τ.

(b) Find the interior of the set $\{\sqrt{2} + n : n \in \mathbb{N}\}$ in $(\mathbb{R}, \tau')$. 3+2

6. (a) Prove that an isometry $f : (X, d) \to (Y, d')$ is a homeomorphism from $(X, \tau(d))$ to $(Y, \tau(d'))$. (Here (X, d) and (Y, d') are two metric spaces and $\tau(d)$ and $\tau(d')$ are the topologies generated by the corresponding metric on X and Y respectively.)

(b) If $\{A_\alpha : \alpha \in \Lambda\}$ is an infinite family of subsets in any topological space (X, τ), then the equality $\overline{\bigcup_{\alpha \in \Lambda} A_\alpha} = \bigcup_{\alpha \in \Lambda} \bar{A}_\alpha$ is always true— correct or justify. 3+2

7. (X, τ) is a topological space and D is a dense subset of X.

(a) Prove that, for an open subset Y of X, $D \cap Y$ is dense in the subspace topology on Y. Is the result true if Y is not open? Justify.

(b) Prove that for a continuous surjection $f : (X, \tau) \to (Z, \tau')$ the set $f(D)$ is dense in Z, where (Z, τ') is any topological space. 3+2

8. If (X, τ) is a second countable space and B is a base for τ, then prove that there exists a countable subfamily D of B such that D is a base for τ. 5

Unit - 2

(Marks : 10)

Answer ***any two*** questions.

9. Let $f: X \to Y$ be any function from a topological space X into a topological space Y. If f is continuous, then prove that the graph of f defined by $G(f)=\{(X, f(x)): x \in X\}$ is homeomorphic to X. 5

10. (a) Prove that a topological space (X, τ) is Hausdorff if the diagonal $\{(x, x) : x \in X\}$ is a closed set in the product space $(X \times X,\ \tau \times \tau)$.

(b) Prove or disprove : In a topological space (X, τ), if every covergent sequence in X has unique limit then X is a T_2 space. 3+2

11. (a) $f: (X, \tau) \to (Y, \tau')$ is an open, continuous, surjection and (X, τ) is a first countable space. Prove that Y is first countable.

(b) Consider a topology η on $\mathbb{R}$ given by $\eta = \{U \subseteq \mathbb{R}:\ \text{either } 1 \notin U \text{ or } \mathbb{R} \backslash U \text{ is finite}\}$. Prove that $(\mathbb{R}, \eta)$ is not first countable. 3+2

12. (a) $f: (X, \tau) \to (Y, \tau')$ is a continuous and injective, where Y is a Hausdorff space. Show that X is Hausdorff.

(b) If (X, τ) is a T_1 space and every intersection of open sets is open in (X, τ), prove that τ is the discrete topology on X. 3+2

Unit - 3

(Marks : 15)

Answer ***any three*** questions.

13. (a) Prove or disprove : The intersection of any family of compact subsets of a space is compact.

(b) Prove or disprove : $(\mathbb{R}, \tau_c)$ is a compact space, where $\tau_c = \{U \subseteq \mathbb{R}:\ \text{either } \mathbb{R} \backslash U \text{ is countable or } \mathbb{R}\}$ 3+2

14. (a) A and B are two compact subsets of a space (X, τ) such that each point of A is strongly separated from each point of B. Prove that A and B are strongly separated in X.

(b) 'There does not exist a continuous map from [2, 5] onto (1, 4), where [2, 5] and (1, 4) are endowed with the subspace topology of the usual topology on $\mathbb{R}$'— Justify the statement. 3+2

15. (a) In a topological space (X, τ), E is a connected subset of X so that $E = A \cup B \cup C$, where A and B are separated and C is connected. Show that $A \cup C$ is connected.

(b) Consider $\mathbb{R}$ endowed with the usual topology, $f: \mathbb{R} \to \mathbb{R}$ is any function such that $f(\mathbb{Q}) \subseteq \mathbb{R} \setminus \mathbb{Q}$ and $f(\mathbb{R} \backslash \mathbb{Q}) \subseteq \mathbb{Q}$. Show that f is not a continuous function. 3+2

16. (a) If every real valued continuous function defined on a topological space X takes on every value between any two values that it assumes then prove that X is connected.

(b) Prove that a continuous mapping from a connected space to the real line having only rational values is constant. 3+2

17. (a) If A is a connected subset of a metric space (X, d) consisting of atleast two points then prove that A is uncountable.

(b) Find all components of the set of rational numbers endowed with the subspace topology from the usual topology of $\mathbb{R}$. 3+2