2020

MATHEMATICS — HONOURS

Paper : CC-5

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$ $\mathbb{Q}$ *denote the sets of real and rational numbers respectively.*

Group - A

(Marks : 20)

1. Answer the following multiple choice questions having only one correct option. Choose the correct option and justify : (1+1)×10

(a) $\lim_{x\to 0} \dfrac{1}{1+e^{\frac{1}{x}}} =$

(i) 0 (ii) 1 (iii) $\frac{1}{2}$ (iv) Does not exist.

(b) If $\lim_{x\to a} |f(x)| = |\ell|$, then $\lim_{x\to a} f(x) =$

(i) ℓ (ii) $-\ell$ (iii) nothing can be said (iv) $\pm\ell$.

(c) If $f(x)$ is continuous on the closed interval [3, 4], then in [3, 4]

(i) $|f(x)| \le M \ \forall x \in [3,4]$ where $M > 0$.

(ii) f is constant.

(iii) f is monotonic increasing.

(iv) f is monotonic decreasing.

(d) $f(x) = 1 - x, \quad x > 0$
$= 2 + x, \quad x < 0$
$= 1, \quad x = 0$

(i) At $x = 0$, f is continuous.

(ii) At $x = 0$, f is left continuous.

(iii) At $x = 0$, f is right continuous.

(iv) $\underset{x\to 0}{Lt} f(x)$ does not exist.

(e) $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \begin{cases} \left|\sin \frac{1}{x}\right|, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then the oscillation of 'f' at $x = 0$ is

(i) -1 (ii) 0

(iii) 1 (iv) 2.

(f) Let $f(x) = \min\{x, x^2\}, x \in \mathbb{R}$. Then identify the correct statement.

(i) f is differentiable on R.

(ii) f is nowhere differentiable.

(iii) f is not differentiable at two values of x.

(iv) f is continuous except at two points.

(g) If $y = x^4 + 2x^3 - 3x^2 - 4x$, then which one of the following statements is correct?

(i) y is increasing for $x < -2$

(ii) y is increasing for $-2 < x < -\frac{1}{2}$

(iii) y is decreasing for $-\frac{1}{2} < x < 1$

(iv) y is decreasing for $x > 1$.

(h) Which one of the following statements is correct for the function $f(x) = x^3$?

(i) f has a local maximum at $x = 0$.

(ii) f has a local minimum at $x = 0$.

(iii) f has neither local maximum nor local minimum at $x = 0$.

(iv) f has local extremum at a point in $\mathbb{R} - \{0\}$.

(i) Which one of the following functions is uniformly continuous on the domain as stated?

(i) $f(x) = x^2, \quad x \in \mathbb{R}$ (ii) $f(x) = \frac{1}{x}, \ x \in [1, \infty)$

(iii) $f(x) = \tan x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (iv) $f(x) = [x], \ x \in [0, 1]$.

(j) Which of the functions does not satisfy the conditions of Rolle's theorem in $[-1, 1]$?

(i) x^2 (ii) $\frac{1}{x^2}$

(iii) $\frac{1}{x^2 + 4}$ (iv) $\sqrt{x^2 + 3}$.

Group - B

(Marks : 25)

Answer ***any five*** questions.

2. (a) Let a function $f: \mathbb{R} \to \mathbb{R}$ be defined as follows :

$$f(x) = \begin{cases} 1 & \text{when } x \in \mathbb{Q} \\ -1 & \text{when } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Using sequential criterion of limit of a function, show that $\lim_{x \to a} f(x)$ does not exist ($a \in \mathbb{R}$).

(b) Give an example of a function f defined over an interval I, such that

(i) f has jump discontinuity at a point of I.

(ii) f has removable discontinuity at a point of I. 3+2

3. (a) Evaluate : $\lim_{x \to 0+} \left(\frac{1 + \tan x}{1 + \sin x} \right)^{\frac{1}{\sin x}}$.

(b) Let $f: [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$ and f assumes only rational values.

If $f\left(\frac{1}{2}\right) = \frac{1}{2}$, prove that $f(x) = \frac{1}{2}$ for all $x \in [0, 1]$. 3+2

4. Show that the image of a closed and bounded interval under a real-valued continuous function f is a closed and bounded interval. 5

5. (a) Prove or disprove : If $f, g : \mathbb{R} \to \mathbb{R}$ be such that fog is continuous at $a \in \mathbb{R}$, then both f and g are continuous at 'a'.

(b) Let $f(x) = x - [x], x \in \mathbb{R}$ where $[x]$ denotes the largest integer not exceeding x. Determine the discontinuities of f and show that they are all of the first kind. 2+3

6. If $f: [a, b] \to \mathbb{R}$ be strictly monotonic and continuous on $[a, b]$, prove that f admits of an inverse function, which is monotonic and continuous on $f([a, b])$. 2+3

7. (a) Applying Sandwich theorem; evaluate $\lim_{x \to 0} \frac{\sin x}{x}$.

(b) If f is a real-valued continuous function on $[a, b]$, then prove that f is uniformly continuous on $[a, b]$. 2+3

8. (a) Let f be continuous in an interval I and does not vanish anywhere in I. Show that f assumes the same sign throughout I.

(b) Give example of a function which is continuous on $\mathbb{R}$, attains its supremum but is not bounded below. 3+2

9. (a) Prove that there exist a point $a \in \left(0, \pi/2\right)$ such that $a = \cos a$.

(b) Show that $\lim_{x \to \infty} a^x \cdot \sin \frac{b}{a^x} = \begin{cases} 0 & \text{if } 0 < a < 1 \\ b & \text{if } a > 1 \end{cases}$. 2+3

Group - C

(Marks : 20)

Answer ***any four*** questions.

10. Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and $f'(a) < f'(b)$. Prove that $f'(x)$ assumes every value between $f'(a)$ and $f'(b)$. 5

11. State and prove Cauchy's Mean Value theorem. 1+4

12. (a) Prove that $f(3)$ is a local minimum value of $f(x) = |3 - x| + |2 + x| + |5 - x|, x \in \mathbb{R}$ but $f'(3)$ does not exist.

(b) Evaluate : $\lim_{x \to 1-} (1 - x)^{\cos \frac{\pi x}{2}}$. 3+2

13. If $f(x) = \begin{cases} \sin x \times \sin\left(\frac{1}{\sin x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

Show that f is continuous at $x = 0$ but not derivable at that point. 2+3

14. (a) Let $f : [0, 2] \to \mathbb{R}$ be a differentiable function such that $f(0) = 0, f(1) = 2, f(2) = 1$.

Prove that $f'(c) = 0$ for some $c \in (0, 2)$.

(b) Expand e^x as an infinite series $(x \in \mathbb{R})$. 2+3

15. Given $f^{n+1}(x)$ is continuous at $x = a$ and $f^{n+1}(a) \neq 0$.

Show that $\lim_{h \to 0} \theta = \frac{1}{n+1}$, where θ is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h).$$ 5

16. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the metal will be the least when the depth of the tank is half of the side of the base. 5
