2020

MATHEMATICS — HONOURS

Paper : DSE-A-2

(Advanced Algebra)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[*Notations Have Usual Meanings*]

Group - A

(Marks : 20)

1. Answer ***all*** questions. In each question *one* mark is reserved for selecting the correct option and *one* mark is reserved for justification. (1+1)×10

(a) The number of Sylow 2-subgroups of A_4 is

(i) 4 (ii) 1 (iii) 3 (iv) 2.

(b) Let G be a group of order 15. Then the centre of G is isomorphic to

(i) $(\mathbb{Z}_3, +)$ (ii) $(\mathbb{Z}_5, +)$ (iii) $(\mathbb{Z}_7, +)$ (iv) $(\mathbb{Z}_{15}, +)$.

(c) Which of the following is a simple group?

(i) $(\mathbb{Z}, +)$ (ii) $(\mathbb{Q}, +)$ (iii) $(\mathbb{Z}_{16}, +)$ (iv) $(\mathbb{Z}_{37}, +)$.

(d) Let G be a finite group. Which of the following statements is true?

(i) G is isomorphic to a cyclic subgroup of S_n for some positive integer n.

(ii) G is isomorphic to a subgroup of A_n for some positive integer n.

(iii) $G = S_n$ for some positive intger n.

(iv) G is isomorphic to $\mathbb{Z}_n$ for some positive integer n.

(e) Which one of the following statements is false?

(i) $x^2 - 2$ is irreducible in $\mathbb{Z}[x]$

(ii) $3x + 6$ is irreducible in both $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$

(iii) $x^2 - 2$ is irreducible in $\mathbb{Q}[x]$ but not so in $\mathbb{R}[x]$

(iv) $\mathbb{Z}_2[x]$ is not an infinite field.

(f) Which one of the following rings is not a regular ring?

(i) $(\mathbb{R}, +, \cdot)$ (ii) $(\mathbb{Q}, +, \cdot)$ (iii) $(\mathbb{Z}, +, \cdot)$ (iv) $(\mathbb{Z}_6, +, \cdot)$.

(g) Identify the correct statement.

(i) $\mathbb{Z}\left[\sqrt{5}\right]$ is a principal ideal domain.

(ii) $\mathbb{Z}\left[\sqrt{5}\right]$ is a Euclidean domain.

(iii) $\mathbb{Z}\left[\sqrt{2}\right]$ is a Euclidean domain.

(iv) $\mathbb{Z}\left[\sqrt{2}\right]$ is not a Euclidean domain.

(h) Let R be a commutative ring with unity. Find which one of the following statements is true.

(i) Every ideal of R is a prime ideal.

(ii) Every ideal of R is a principal ideal.

(iii) Every ideal of R is a maximal ideal.

(iv) Every maximal ideal of R is a prime ideal.

(i) The member of solutions of the polynomial equation $x^2 + x = 0$ in $\mathbb{Z}_6$ is

(i) 2 (ii) 4 (iii) 6 (iv) none of these.

(j) All the associates of [6] in $\mathbb{Z}_{10}$ are

(i) [2], [4], [7], [9]

(ii) [3], [5], [7], [8]

(iii) [2], [4], [6], [8]

(iv) [2], [4], [6], [9].

Group - B

(Marks : 15)

2. Answer ***any three*** questions :

(a) (i) Consider the alternating group A_3 on the set $S = \{1, 2, 3\}$. Prove that there exists a group action of A_3 on S.

(ii) Prove that every group of order p^2 where p is a prime, is commutative. 3+2

(b) (i) If G is a group of order p^n where p is a prime and n is a positive integer, then show that the centre $Z(G) \neq \{e\}$ where 'e' is the identity element of G.

(ii) Prove or disprove : There are 6 elements of order 7 in a group of order 28. 3+2

(c) State and prove Sylow's First Theorem. 1+4

(d) (i) Prove that $6 = 1 + 2 + 3$ is a class equation of a finite group.

(ii) Prove that for any group G, $\left| G/Z(G) \right| \neq 51$. 3+2

(e) Show that A_5 is a simple group. 5

Group - C

(Marks : 30)

3. Answer ***any six*** questions :

(a) (i) Using Eisenstein's criterion, prove that the polynomial $10x^3 - 7x + 14$ is irreducible over $\mathbb{Q}$.

(ii) Show that $\mathbb{Z}[x]$ is not a principal ideal domain. 3+2

(b) Define greatest common divisor (*gcd*) of a pair of elements of a ring. Give an example of a ring R and a pair of elements a, $b \in R$ such that $gcd(a, b)$ does not exist. 2+3

(c) (i) Find $gcd\,(2 - 7i, 2 + 11i)$ in the ring of Gaussian integers $\mathbb{Z}[i]$.

(ii) In an integral domain R, prove that two elements a and b of R are associate with each other if and only if $\langle a \rangle = \langle b \rangle$. 3+2

(d) (i) Let $\omega = \frac{-1+\sqrt{-3}}{2}$ and $\mathbb{Z}[\omega] = \{r + s\omega \mid r, s \in \mathbb{Z}\}$. Prove that $\mathbb{Z}[\omega]$ is a Euclidean domain.

(ii) Let R be a Euclidean domain with Euclidean valuation δ and $u \in R$. If $\delta(u) = \delta(1)$, prove that u is a unit in R. 3+2

(e) (i) In $(\mathbb{Z}_{12}, +, \cdot)$, prove that the element [3] is prime but not irreducible.

(ii) In the ring of Gaussian integers $\mathbb{Z}[i]$, show that the element 5 is not irreducible. 3+2

(f) (i) Prove that isomorphic integral domains have isomorphic quotient fields.

(ii) Find all irreducible polynomials of degree 2 over the field $\mathbb{Z}_3$. 3+2

(g) (i) Prove that a factorization domain D is a unique factorization domain if and only if every irreducible element of D is prime.

(ii) Prove that the elements $[4]x + [1]$ and $[2]x + 3$ are units in the ring $\mathbb{Z}_8[x]$. 3+2

(h) Show that every integral domain can be embedded in a field. 5

(i) Prove that the centre of a regular ring is again regular. 5

(j) Determine all the prime elements of the ring $\mathbb{Z}[i]$. 5

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