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INTUITIONISTIC FUZZY ALMOST COMPACT SETS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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Abstract. This paper is aimed at introducing and discussing the concept of intuitionistic fuzzy almost compact set in intuitionistic fuzzy topological spaces. Characterizations and properties of these sets, are established using the notion of intuitionistic fuzzy θ-cluster point of intuitionistic fuzzy filterbases and intuitionistic fuzzy θ-closure operator.

1. Introduction and Preliminaries

It is found in the literature that compactness is being treated as a very powerful notion in classical set topology for a long time. Zadeh [9] initiated the concept of fuzzy set, and later on Atanassov [2] generalized it into another parallel framework and called it intuitionistic fuzzy set. Since then researchers were very much interested to extend compactness to fuzzy and intuitionistic fuzzy fields.

In fuzzy setting, Chang [3] established fuzzy compactness and since then investigations were going on in the area of different weaker forms of fuzzy compactness viz. fuzzy paracompactness, fuzzy almost compactness, fuzzy near compactness etc. It was Di Concilio and Gerla [6] who defined fuzzy almost compactness. In [7] Mukherjee and Sinha investigated almost compactness for fuzzy topological spaces and for arbitrary fuzzy sets by means of fuzzy filterbases, fuzzy θ-closure and fuzzy θ-cluster points which brought forth a number of characterizations and properties analogous to those in classical set topology. Motivated by these, intuitionistic fuzzy almost compactness in intuitionistic fuzzy topological spaces was investigated in [8]. In the aforementioned paper [8], a satisfactory definition of intuitionistic fuzzy almost compact space was given and its properties with respect to intuitionistic fuzzy regularity, intuitionistic fuzzy θ-closure operator, intuitionistic fuzzy nets and filterbases, interiorly finite intersection property etc. were discussed.

In our present paper, we have turned our attention to the description of intuitionistic fuzzy almost compact sets in an intuitionistic fuzzy topological space X. For developing this concept, in Section 2 we define the notion of intuitionistic fuzzy open cover and its finite proximate subcover for an intuitionistic fuzzy set and after defining intuitionistic fuzzy almost compact sets, we try to identify some intuitionistic fuzzy almost compact sets in X. After that we discuss about intuitionistic fuzzy almost compact sets in terms of intuitionistic fuzzy filterbases. In Section 3, we characterize intuitionistic fuzzy almost compact sets in terms of intuitionistic fuzzy θ-closure operator.

In what follows, 'I-fuzzy' will stand for 'Intuitionistic fuzzy' and X, I will respectively denote a nonempty fixed set and the closed interval $[0, 1]$. Before we proceed further, we recall some

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concepts and results used in the paper(for more details, the well known works [4, 5] may be referred too).

Definition 1.1. [2] An object A given by the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ is called an intuitionistic fuzzy set or simply an IFS, where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote respectively the degree of membership and the degree of non-membership of each element $x \in X$ to the set A with $0 \le \mu_A(x) + \gamma_A(x) \le 1$, for each $x \in X$. For brevity, such an IFS A is written as $A = \langle x, \mu_A, \gamma_A\rangle$ instead of $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.2. [2] Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$ be two IFS's and $\{U_j : j \in \Lambda\}$ be any family of IFS's in a nonempty set X. Then we define as follows:

(a) $A \subseteq B$ if $\mu_A(x) \le \mu_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)$, $\forall x \in X$. We shall write '$A(x) \le B(x)$, for all $x \in X$' to denote "$A \subseteq B$". For a particular $y \in X$, we shall write '$A(y) \le B(y)$' or $\langle y, \mu_A(y), \gamma_A(y)\rangle \le \langle y, \mu_B(y), \gamma_B(y)\rangle$ if $\mu_A(y) \le \mu_B(y)$ and $\gamma_A(y) \ge \gamma_B(y)$.

(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(c) $\overline{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$.

(d) $\bigcap\limits_{j\in\Lambda} U_j = \{\langle x, \bigwedge \mu_{U_j}(x), \bigvee \gamma_{U_j}(x)\rangle : x \in X\}$ and $\bigcup\limits_{j\in\Lambda} U_j = \{\langle x, \bigvee \mu_{U_j}(x), \bigwedge \gamma_{U_j}(x)\rangle : x \in X\}$, where, as usual, $\bigvee \mu_{U_j}(x) = \sup\{\mu_{U_j}(x) : j \in \Lambda\}$ and $\bigwedge \gamma_{U_j}(x) = \inf\{\gamma_{U_j}(x) : j \in \Lambda\}$for $x \in X$.

(e) $1_\sim = \{\langle x, 1, 0\rangle : x \in X\}$ and $0_\sim = \{\langle x, 0, 1\rangle : x \in X\}$.

It is clear that $\overline{\overline{U}} = U$, $\overline{0}_\sim = 1_\sim$ and $\overline{1}_\sim = 0_\sim$.

Proposition 1.1. [8] *If $\{A_\alpha : \alpha \in \Lambda\}$ is an arbitrary collection of IFS's in X, then* $\overline{\bigcup\limits_{\alpha}\{A_\alpha : \alpha \in \Lambda\}} = \bigcap\limits_{\alpha}\{\overline{A_\alpha} : \alpha \in \Lambda\}$ *and* $\overline{\bigcap\limits_{\alpha}\{A_\alpha : \alpha \in \Lambda\}} = \bigcup\limits_{\alpha}\{\overline{A_\alpha} : \alpha \in \Lambda\}$.

Definition 1.3. [4] A family τ of IFS's in X is called an intuitionistic fuzzy topology (or simply an IFT) on X if the following conditions are satisfied:

(i) $0_\sim, 1_\sim \in \tau$

(ii) $U_1, U_2 \in \tau \Rightarrow U_1 \bigcap U_2 \in \tau$

(iii) $\{U_j : j \in \Lambda\} \subseteq \tau \Rightarrow \bigcup\limits_{j\in\Lambda} U_j \in \tau$.

The pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS, for brevity) and each member of τ is an intuitionistic fuzzy open set (IFOS, in short) in X. An intuitionistic fuzzy closed set (abbreviated as IFCS) is defined to be the complement of an IFOS in (X, τ).

Definition 1.4. [4] The I-fuzzy interior and I-fuzzy closure of an IFS A in an IFTS (X, τ) are defined as follows: $cl(A) = \bigcap\{B : B$ is an IFCS in X and $A \subseteq B\}$ and $int(A) = \bigcup\{U : U$ is an IFOS in X and $U \subseteq A\}$.

It is known [8] that for any IFS A in (X, τ), $cl(A)$ and int(A) are respectively an IFCS and an IFOS; and also, A is an IFCS (IFOS) in $X \Leftrightarrow cl(A) = A$ (resp $int(A) = A$).

Proposition 1.2. [4] *For any two IFS's A and B in an IFTS (X, τ), the following results hold:*

(a) $A \subseteq cl(A)$ *and* $int(A) \subseteq A$

(b) $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$ *and* $int(A) \subseteq int(B)$

(c) $cl(clA) = clA$ *and* $int(intA) = intA$

(d) $cl(A \cup B) = clA \cup clB$ *and* $int(A \cap B) = intA \cap intB$

(e) $cl(\overline{A}) = \overline{int(A)}$ *and* $int(\overline{A}) = \overline{cl(A)}$.

Result 1.3. [8] For any family $\{A_\alpha : \alpha \in \Lambda\}$ of IFS's in an IFTS X,
(i) $\cup cl(A_\alpha) \subseteq cl(\cup A_\alpha)$ and the equality holds if Λ is finite;
(ii)$\cup int(A_\alpha) \subseteq int(\cup A_\alpha)$.

Definition 1.5. [4] A collection $\{\langle x, \mu_{A_i}, \gamma_{A_i}\rangle : i \in \Lambda\}$ of IFOS's in an IFTS (X, τ) such that $\bigcup_{i\in\Lambda} \langle x, \mu_{A_i}, \gamma_{A_i}\rangle = 1_\sim$ is called an I-fuzzy open cover of X.

Definition 1.6. Let (X, τ) be an IFTS. A finite subcollection $\mathcal{U}_0 = \{\langle x, \mu_{A_i}, \gamma_{A_i}\rangle : i = 1, 2, ..., n\}$ of an I-fuzzy open cover $\mathcal{U}$ of X is called
(a) a finite subcover [4] of $\mathcal{U}$ if $\bigcup \mathcal{U}_0 = 1_\sim$;
(b) a finite proximate subcover [8] of $\mathcal{U}$ if $\bigcup_{i=1}^{n}(cl\langle x, \mu_{A_i}, \gamma_{A_i}\rangle) = 1_\sim$.

Definition 1.7. [4] An IFTS X in which every I-fuzzy open cover of X has a finite subcover, is called an *I*-fuzzy compact space.

Definition 1.8. [4] Let p be a chosen point of X and $a \in (0, 1]$, $b \in [0, 1)$ be two chosen real numbers with $a + b \leq 1$, then the IFS $\langle x, p_a, 1 - p_{1-b}\rangle$, denoted by $p(a, b)$, is called an intuitionistic fuzzy point (IFP, for brevity) in X, where a stands for the degree of membership of $p(a, b)$, the support of $p(a, b)$ being p.

Definition 1.9. [8] An IFP $p(a, b)$, where $a, b \in (0, 1)$ is said to be contained in an IFS $U = \langle x, \mu_U, \gamma_U\rangle$ in X if $a \leq \mu_U(p)$ and $b \geq \gamma_U(p)$. In this case we write $p(a, b) \leq U$.

Proposition 1.4. [8] *Any IFS A in* X *coincides with the union of all those IFP's that are contained in A.*

Definition 1.10. [5] Given an IFP $p(a, b)$ and an IFS $V = \langle x, \mu_V, \gamma_V\rangle$, $p(a, b)$ is said to be quasi-coincident with V (in notation, $p(a, b)qV$) if $a > \gamma_V(p)$ or $b < \mu_V(p)$.

Definition 1.11. [8] Two IFS's $U = \langle x, \mu_U, \gamma_U\rangle$ and $V = \langle x, \mu_V, \gamma_V\rangle$ in X are called quasi-coincident, if for some $x \in X$, $\mu_U(x) > \gamma_V(x)$ or $\gamma_U(x) < \mu_V(x)$. In this case we write UqV, while we write "$U\overline{q}V$" to mean that U and V are not quasi-coincident.

Result 1.5. [5] For any two IFS's A and B in X, $A \subseteq B$ holds iff $A\overline{q}\overline{B}$ holds.

Definition 1.12. [8] An IFS A is said to be a q-nbd of an IFP $p(a, b)$ in an IFTS (X, τ) if $p(a, b)qU \subseteq A$ holds for some IFOS U in X.

Result 1.6. [8] Let $p(a, b)$ be an IFP and A an IFS in an IFTS X. Then $p(a, b) \leq cl(A)$ if and only if for each q-nbd U of $p(a, b)$, UqA.

Lemma 1.7. [8] *Let A be an IFS and B an IFOS in an IFTS (X, τ). Then $A\overline{q}B \Rightarrow cl(A)\overline{q}B$.*

2. Intuitionistic fuzzy almost compact set

In this section, we extend the concept of I-fuzzy almost compactness to arbitrary I-fuzzy sets and study its properties. We also characterize intuitionistic fuzzy almost compact sets by using I-fuzzy filterbase.

Definition 2.1. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be an I-fuzzy set in an IFTS X. Then
(a) the set $\{x : x \in X\, and\, \langle x, \mu_A(x), \gamma_A(x)\rangle > 0_\sim(x)\}$ is called the support of A and is denoted by $suppA$; i.e., $x \in SuppA$ iff $\mu_A(x) > 0$ and $\gamma_A(x) < 1$.

(b) a collection $\mathcal{U} = \{U_i = \langle x, \mu_{U_i}(x), \gamma_{U_i}(x)\rangle : i \in \Lambda\}$ of IFS's in X is said to be an I-fuzzy cover of A if $[\bigcup_{i\in\Lambda}\langle x, \mu_{U_i}(x), \gamma_{U_i}(x)\rangle](x) = 1_\sim(x)$ for each $x \in suppA$.

If in addition, the members of $\mathcal{U}$ are I-fuzzy open in X, then $\mathcal{U}$ is called an I-fuzzy open cover of A.

(c) an I-fuzzy open cover $\mathcal{U} = \{U_i = \langle x, \mu_{U_i}(x), \gamma_{U_i}(x)\rangle : i \in \Lambda\}$ of A is said to have a finite subcover $\mathcal{U}_0$ for A if $\mathcal{U}_0$ is a finite sub-collection of $\mathcal{U}$ such that $\mathcal{U}_0 = \{\langle x, \mu_{U_i}, \gamma_{U_i}\rangle : i \in \Lambda_0\}$ (where Λ_0 is a finite subset of Λ) and $[\bigcup_{i\in\Lambda_0}\langle x, \mu_{U_i}, \gamma_{U_i}\rangle](x) \geq \langle x, \mu_A, \gamma_A\rangle(x)$ for each $x \in suppA$.

Definition 2.2. An I-fuzzy set A is called I-fuzzy compact if every I-fuzzy open cover of A has a finite I-fuzzy subcover for A.

Remark 2.1. It is worth mentioning that in the above definitions if we take $A = X$, the concept of I-fuzzy (open) cover and I-fuzzy compactness of A coincide respectively with those for an IFTS X as given by Dogan Coker [4].

Definition 2.3. An I-fuzzy open cover $\mathcal{U} = \{U_i = \langle x, \mu_{U_i}(x), \gamma_{U_i}(x)\rangle : i \in \Lambda\}$ of an I-fuzzy set A in an IFTS X is said to have a finite proximate subcover for A if there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \bigcup_{U_i\in\mathcal{U}_0}\{cl\langle x, \mu_{U_i}, \gamma_{U_i}\rangle\}$.

Remark 2.2. Here also we like to note that if $A = X$, the above definition agrees with that of an open cover of an IFTS X to have a finite proximate subcover as given in [8].

Definition 2.4. An I-fuzzy set A in an IFTS X is said to be an I-fuzzy almost compact set relative to X (IFAC set, for short) if every I-fuzzy open cover of A has a finite I-fuzzy proximate subcover for A.

Proposition 2.1. *Every I-fuzzy compact set in an IFTS is also an I-fuzzy almost compact set relative to X.*

Remark 2.3. That the converse of the above result is false, is shown by the following example.

Example 2.2. Let X be an infinite set and a, b be two arbitrarily chosen points of X, kept fixed. Suppose A is an IFS in X given by $\{\langle x, \mu_x, \gamma_x\rangle : x \in X\}$, where

$$\mu_x = \begin{cases} 1, & for\ x = a, b \\ 0, & for\ x \in X \setminus \{a, b\} \end{cases}$$

$$\gamma_x = \begin{cases} 0, & for\ x = a, b \\ 1/2, & for\ x \in X \setminus \{a, b\} \end{cases}$$

Consider next the IFS A_n, for $n \in \mathbb{N}$, given by $A_n = \{\langle x, \mu_x^n, \gamma_x^n\rangle : x \in X\}$, where

$$\mu_x^n = \begin{cases} 1 - 1/n, & for\ x = a, b \\ 1/2, & for\ x \in X \setminus \{a, b\} \end{cases}$$

$$\gamma_x^n = \begin{cases} 1/n, & for\ x = a, b \\ 1/2, & for\ x \in X \setminus \{a, b\} \end{cases}$$

It is easy to see that $\tau = \{1_\sim, 0_\sim, A_n : n \in \mathbb{N}\} \cup \{\bigcup_{n=1}^{\infty} A_n\}$ is an IFT on X. Now, the collection $\{A_n : n \in \mathbb{N}$ and $n > 2\}$ is an I-fuzzy open cover of A which has no finite subcover. Thus A is not I-fuzzy compact. Here we note that $\{A_n : n \in \mathbb{N}$ and $n > 2\}$ is not an I-fuzzy open cover of the IFTS (X,τ)(it will be so if $X = \{a,b\}$). Also, the IFTS is I-fuzzy compact (as every I-fuzzy open cover of (X,τ) has to include $1_\sim$ as a member of it) and hence IFAC. Next we see that for each $n \in \mathbb{N}$, $clA_n = 1_\sim$. In fact, for any $n \in \mathbb{N} \setminus \{1,2\}$ we have, $\mu_{A_n}(a) > \mu_{\overline{A}_m}(a)$ for all $m > 2$, so that $A_n \not\leq \overline{A_m}$. Thus the only IFCS containing $A_n(n \in \mathbb{N},\ n > 2)$ is $1_\sim$. Consequently, $clA_n = 1_\sim$, for each $n \in \mathbb{N}\ (n > 2)$. Thus for any I-fuzzy open cover $\mathcal{U}$ of A, if we select any $A_n \in \mathcal{U}$ (for some $n > 2$), then $\{A_n\}$ is a finite proximate subcover of $\mathcal{U}$ for A. Thus A is I-fuzzy almost compact.

Before going to the properties of I-fuzzy almost compact sets, our preliminary concern is to identify certain classes of IFAC sets in an intuitionistic fuzzy topological space. To that end we have the following theorem:

Theorem 2.3. *Let X be an IFTS. Then*
(i) $cl(A)$ is an IFAC set in X for any IFAC set A in X.
(ii) Union of any finite collection of IFAC set in X is again an IFAC set in X.

Proof. Straightforward and is omitted.
It is known that almost compactness can be achieved by using the notion of regular open coverings. Our next goal is to get a similar result in I-fuzzy setting. For this we recall: □

Definition 2.5. [8] An IFS A in an IFTS (X,τ) is said to be I-fuzzy regular open(regular closed) if $A = int(clA)$ (respectively $A = cl(intA)$).

Proposition 2.4. *In an IFTS (X,τ)*
(i) the closure of an I-fuzzy open set is an I-fuzzy regular closed set and
(ii) the interior of an I-fuzzy closed set is an I-fuzzy regular open set.

Proof. (i) Let $A = \langle x, \mu_A, \gamma_A\rangle$ be an I-fuzzy open set in an IFTS X. We have, $int(clA) \subseteq clA$ $\Rightarrow clint(clA) \subseteq clA$. On the other hand, A is I-fuzzy open $\Rightarrow A \subseteq int(clA) \Rightarrow clA \subseteq clint(clA)$. Thus the result follows.
(ii) Similar as (i) above. □

Theorem 2.5. *Every I-fuzzy regular closed set A in an IFAC space X is an IFAC set in X.*

Proof. Let $A = \langle x, \mu_A, \gamma_A\rangle$ be an I-fuzzy open set in an IFTS X and let $\mathcal{U} = \{U_i = \langle x, \mu_{U_i}(x), \gamma_{U_i}(x)\rangle : i \in \Lambda\}$ be an I-fuzzy open cover of A. Thus for each $x \in suppA$, $[\bigcup_{i\in\Lambda}\langle x, \mu_{U_i}(x), \gamma_{U_i}(x)\rangle](x) = 1_\sim(x)$. Now if $x \notin suppA$, $\langle x, \mu_A(x), \gamma_A(x)\rangle(x) = 0_\sim(x) \Rightarrow \overline{\langle x, \mu_A(x), \gamma_A(x)\rangle}(x) = 1_\sim(x)$ for each $x \notin suppA$. Thus $[\bigcup \overline{\langle x, \mu_A(x), \gamma_A(x)\rangle}](x) = 1_\sim(x)$ for each $x \notin suppA$. So $\mathcal{U} \cup \{\overline{A}\}$ is an I-fuzzy open cover of X. Since X is IFAC, there exist finitely many sets $U_1, U_2, ..., U_n$ of $\mathcal{U}$ such that $[cl(U_1) \cup cl(U_2) \cup ... \cup cl(U_n)] \cup (cl\overline{A}) = 1_\sim$ $\Rightarrow \overline{cl(U_1) \cup cl(U_2) \cup ... \cup cl(U_n)} \cap (intA) = 0_\sim \Rightarrow (intA)\overline{q}[\overline{cl(U_1) \cup cl(U_2) \cup ... \cup cl(U_n)}] \Rightarrow (intA) \subseteq [cl(U_1) \cup cl(U_2) \cup ... \cup cl(U_n)] \Rightarrow A = cl(intA) \subseteq [cl(U_1) \cup cl(U_2) \cup ... \cup cl(U_n)]$. Hence A becomes an IFAC set. □

Corollary 2.6. *If A is an I-fuzzy open set in an IFAC space X, then $cl(A)$is an IFAC set in X.*

Proof. Follows from Proposition 2.4 and Theorem 2.5. □

Our next intention is to characterize intuitionistic fuzzy almost compact sets by I-fuzzy filterbases. For doing this we need the following:

Definition 2.6. [8] A collection $\mathcal{B}(\neq \phi)$ of IFS's in an IFTS (X, τ) is called an I-fuzzy filterbase in X if
(i) $B \neq 0_X$, $\forall B \in \mathcal{B}$ and
(ii) $B_1, B_2 \in \mathcal{B} \Rightarrow \exists\ B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$.
In addition, if $\mathcal{B} \subseteq \tau$, then $\mathcal{B}$ is called an I-fuzzy open filterbase.
In particular, $\mathcal{B}$ is called an I-fuzzy filterbase in an IFS A, if $(B \in \mathcal{B} \Rightarrow B \subseteq A)$.

Definition 2.7. [8] An IFP $p(a, b)$ in X is defined to be an I-fuzzy cluster point of an I-fuzzy filterbase $\mathcal{B}$ in X if for each $B \in \mathcal{B}$, $p(a, b) \leq cl(B)$.

Theorem 2.7. [8] *A necessary and sufficient condition for an IFTS (X, τ) to be IFAC is that every I-fuzzy open filterbase in X has an I-fuzzy cluster point.*

Theorem 2.8. *An I-fuzzy set A in an IFTS (X, τ) is an IFAC set iff for every I-fuzzy open filterbase $\mathcal{F}$ in X, $[\bigcap\{clF : F \in \mathcal{F}\}] \bigcap A = 0_\sim$ implies that there exists a finite collection $\mathcal{F}_0$ of $\mathcal{F}$ such that $[\bigcap\{F : F \in \mathcal{F}_0\}]\ \overline{q}\ A$.*

Proof. Let $A = \langle x, \mu_A, \gamma_A\rangle$ be an IFAC set in an IFTS X. Suppose $\mathcal{F}$ is an I-fuzzy open filterbase in X such that for each finite subcollection $\mathcal{F}_0$ of $\mathcal{F}$, $[\bigcap\{F : F \in \mathcal{F}_0\}]\ q\ A$ but $[\bigcap\{clF : F \in \mathcal{F}\}] \bigcap A = 0_\sim$. Then for each $x \in suppA$, $[\bigcap\limits_{F\in\mathcal{F}} \langle x, \mu_{clF}, \gamma_{clF}\rangle](x) = 0_\sim(x)$ $\Rightarrow \overline{[\bigcap\limits_{F\in\mathcal{F}} \langle x, \mu_{clF}, \gamma_{clF}\rangle]}(x) = \overline{0_\sim}(x) \Rightarrow [\bigcup\limits_{F\in\mathcal{F}} \langle x, \mu_{\overline{clF}}, \gamma_{\overline{clF}}\rangle](x) = 1_\sim(x)$. Thus the collection $\{\overline{clF} : F \in \mathcal{F}\} = \mathcal{F}_1$(say), is an I-fuzzy open cover of A. Since A is an IFAC set, there exists a finite proximate subcover $\{\overline{cl(F_1)}, \overline{cl(F_2)}, ..., \overline{cl(F_n)}\}$ (say) of $\mathcal{F}_1$ for A. Then $A \subseteq \bigcup\limits_{i=1}^{n}\{cl(\overline{cl(F_i)})\} \Rightarrow A \subseteq \bigcup\limits_{i=1}^{n}[cl\{\overline{cl\langle x, \mu_{F_i}, \gamma_{F_i}\rangle} : x \in suppA\}] \Rightarrow \overline{\bigcap\limits_{i=1}^{n}[cl\{\overline{cl\langle x, \mu_{F_i}, \gamma_{F_i}\rangle}\}]} \subseteq \overline{A} \Rightarrow$ $\bigcap\limits_{i=1}^{n}[int\{cl\langle x, \mu_{F_i}, \gamma_{F_i}\rangle : x \in suppA\}] \subseteq \overline{A} \Rightarrow$ (by Proposition 1.2 (a)) $\Rightarrow \bigcap\limits_{i=1}^{n}\langle x, \mu_{F_i}, \gamma_{F_i}\rangle \subseteq \overline{A}$ (since F_i's are I-fuzzy open) $\Rightarrow \bigcap\limits_{i=1}^{n}\{\langle x, \mu_{F_i}, \gamma_{F_i}\rangle : x \in suppA\}\ \overline{q}\ A$ (by Result 1.5) which contradicts our assumption.

Conversely, let the given condition hold and if possible, the I-fuzzy set A be not an IFAC set in X. Then there exists an I-fuzzy open cover $\mathcal{U}$ of A which has no finite I-fuzzy proximate subcover for A. Then for every finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$, there exists some $y \in suppA$ such that $\bigcup\limits_{U\in\mathcal{U}_0} \{cl\langle x, \mu_U, \gamma_U\rangle\}(y) < \langle x, \mu_A, \gamma_A\rangle(y) \Rightarrow \bigcap\limits_{U\in\mathcal{U}_0} \overline{\{cl\langle x, \mu_U, \gamma_U\rangle\}}(y) >$ $\overline{\langle x, \mu_A, \gamma_A\rangle}(y) \geq 0_\sim(y)$. Thus the collection $\{\bigcap\limits_{U\in\mathcal{U}_0} \overline{cl\langle x, \mu_U, \gamma_U\rangle} : \mathcal{U}_0$ is a finite subset of $\mathcal{U}\}$ $= \mathcal{V}$ (say) forms an I-fuzzy open filterbase on X. We assert that for every finite subcollection $\mathcal{V}_0$ of $\mathcal{V}$, $(\bigcap\limits_{V\in\mathcal{V}_0} V)\ q\ A$. If not, let there exist a finite subcollection $\{\overline{cl(V_1)}, \overline{cl(V_2)}, ...\overline{cl(V_n)}\}$ of $\mathcal{V}$ such that $\bigcap\limits_{i=1}^{n}\{\overline{cl(V_i)}\}\ \overline{q}\ A$ holds. Then $A \subseteq \overline{\bigcap\limits_{i=1}^{n}\overline{cl(V_i)}}$(by Result 1.5) $= \bigcup\limits_{i=1}^{n} cl(V_i)$. Thus $\mathcal{U}$ has a finite I-fuzzy proximate subcover for A which contradicts our assumption. So by the given condition $\bigcap\{cl(\overline{cl(U)}) : U \in \mathcal{U}\} \bigcap A \neq 0_\sim \Rightarrow \exists y \in suppA$ such that $\bigcap\limits_{U\in\mathcal{U}} [cl\{\overline{cl\langle x, \mu_U, \gamma_U\rangle}\}](y) > 0_\sim(y) \Rightarrow \bigcup\limits_{U\in\mathcal{U}} \overline{[cl\{\overline{cl\langle x, \mu_U, \gamma_U\rangle}\}]}(y) < 1_\sim(y)$ for some $y \in suppA$

$\Rightarrow \bigcup_{U\in\mathcal{U}} [int\{cl\langle x,\mu_U,\gamma_U\rangle\}](y) < 1_\sim(y)$ for some $y \in suppA$. Thus $\bigcup_{U\in\mathcal{U}} \langle x,\mu_U,\gamma_U\rangle(y) < 1_\sim(y)$ for some $y \in suppA$. (since U is I-fuzzy open, $U \subseteq int(clU)$) which contradicts the fact that $\mathcal{U}$ is an I-fuzzy open cover of A. □

Theorem 2.9. *An I-fuzzy set A in an IFTS (X,τ) is an IFAC set iff for any family $\mathcal{G}$ of I-fuzzy closed sets in X with $\bigcap_{G\in\mathcal{G}} [\{G : G \in \mathcal{G}\}] \bigcap A = 0_\sim$ there exists a finite subcollection $\mathcal{G}_0$ of $\mathcal{G}$ such that $\bigcap\{intG : G \in \mathcal{G}_0\}]\ \overline{q}\ A$.*

Proof. Let $A = \langle x,\mu_A,\gamma_A\rangle$ be an IFAC set in an IFTS X. Let $\mathcal{G} = \{\langle x,\mu_G,\gamma_G\rangle : x \in X\}$ be a family of I-fuzzy closed sets in X such that $\bigcap_{G\in\mathcal{G}} \{\langle x,\mu_G,\gamma_G\rangle : x \in X\} \bigcap A = 0_\sim$. Then for each $a \in suppA$, $\bigcap_{G\in\mathcal{G}} \{\langle x,\mu_G,\gamma_G\rangle : x \in X\}(a) = 0 \Rightarrow \bigcup_{G\in\mathcal{G}} \overline{\{\langle x,\mu_G,\gamma_G\rangle : x \in X\}}(a) = 1$ for each $a \in suppA$. Consequently $\{\overline{G} : G \in \mathcal{G}_0\}$ is an I-fuzzy open cover of A and since A is IFAC, there exists a finite I-fuzzy proximate subcover $\mathcal{G}_0$(say) of $\mathcal{G}$ for A. Thus $A \subseteq \bigcup_{G\in\mathcal{G}_0} \{cl\overline{G}\} \Rightarrow \overline{A} \supseteq \overline{\bigcup_{G\in\mathcal{G}_0} \{cl\overline{G}\}} = \bigcap_{G\in\mathcal{G}_0} intG \Rightarrow (\bigcap_{G\in\mathcal{G}_0} intG)\ \overline{q}\ A$ and thus the necessity part is proved.

For sufficiency, suppose $\mathcal{B}$ is an I-fuzzy open filterbase in X such that $\bigcap\{clB : B \in \mathcal{B}\} \bigcap A = 0_\sim$. Then $\mathcal{F} = \{clB : B \in \mathcal{B}\}$ is a family of I-fuzzy closed sets in X with $((\bigcap \mathcal{F}) \bigcap A) = 0_\sim$. Hence according to our assumption, there exists a finite subcollection $\mathcal{B}_0$ of $\mathcal{B}$ such that $\bigcap\{intclB : B \in \mathcal{B}_0\}\ \overline{q}\ A$. Thus $(\bigcap \mathcal{B}_0)\ \overline{q}\ A$ and hence by Theorem 2.8, it follows that A is an IFAC set. □

3. IFAC sets in terms of IF-θ-closure operator

The idea of θ-closure operator was first introduced in classical set topology to act as an useful tool for studying various set topological concepts. Due to its varied applicability, this idea was extended to fuzzy setting by Mukherjee and Sinha in [7]. The said operator was also used in [8], and now our intention is to characterize and study IFAC sets by using the same concept once again. For this we require the following:

Definition 3.1. [8] An IFP $p(a,b)$ in an IFTS X is called an I-fuzzy θ-cluster point (IFθ-cluster point, for brevity) of an I-fuzzy set A if for each I-fuzzy open q-nbd U of $p(a,b)$, $clUqA$. The I-fuzzy θ-closure of A, write as $cl_\theta(A)$, is defined as the union of all IFθ-cluster points of A. An IFS U is said to be IFθ-closed if $U = cl_\theta(U)$, while an IFθ-open set is one whose complement is an IFθ-closed set.

Theorem 3.1. [8] *For any IFO set A in an IFTS (X,τ), $clA = cl_\theta(A)$.*

Definition 3.2. [8] Let $\mathcal{B}$ be an I-fuzzy filterbase in an IFTS X. An IFP $c(a,b)$ is called an I-fuzzy θ-cluster point of $\mathcal{B}$ if $c(a,b) \le \bigcap\{cl_\theta(B) : B \in \mathcal{B}\}$. If in addition, $c(a,b) \le A$ for some IFS A in X, then $\mathcal{B}$ is said to have an I-fuzzy θ-cluster point in A.

Theorem 3.2. [8] *A necessary and sufficient condition for an IFTS X to be I-fuzzy almost compact is that each I-fuzzy filterbase on X has an I-fuzzy θ-cluster point.*

Theorem 3.3. *An I-fuzzy set A in an IFTS (X,τ) is an IFAC set iff for each I-fuzzy filterbase $\mathcal{F}$ in X with the condition that for each finite subcollection $\{F_1,F_2,...,F_n\}$ of $\mathcal{F}$ and for any I-fuzzy regular closed set C containing A, one has $\bigcap_{i=1}^{n} F_i\ q\ C$; $\mathcal{F}$ has an I-fuzzy θ-cluster point in A.*

Proof. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an IFAC set in an IFTS X. Suppose $\mathcal{F}$ be an I-fuzzy filterbase in X without any I-fuzzy θ-cluster point in A. So $A \bigcap [\bigcap \{cl_\theta F : F \in \mathcal{F}\}] = 0_\sim$ ....(1)

Let $p \in suppA$. Then $\mu_A(p) > 0$ and $\gamma_A(p) < 1$. Thus there exists $m \in \mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers) such that $\mu_A(p) > \frac{1}{m}$ and $\gamma_A(p) < 1 - \frac{1}{m}$. i.e., the I-fuzzy point $p(\frac{1}{n}, 1 - \frac{1}{n})$ is contained in A, for all $n \geq m$. Then in view of (1), there exist an I-fuzzy open q-nbd U_p^n of $p(\frac{1}{n}, 1 - \frac{1}{n})$ and an $F_p^n \in \mathcal{F}$ such that $cl(U_p^n)\ \overline{q}\ F_p^n$ ...(2)

Now since U_p^n is a q-nbd of $p(\frac{1}{n}, 1 - \frac{1}{n})$, then $\mu_{U_p^n}(p) > 1 - \frac{1}{n}$ or $\gamma_{U_p^n}(p) < \frac{1}{n}$, for all $n \geq m$...(3)

It is then clear that $inf\{\gamma_{U_p^n}(p) : n \geq m\} = 0$ or $sup\{\mu_{U_p^n}(p) : n \geq m\} = 1$ ... (4)

Now if we put $U = \bigcup_{n \geq m} U_p^n = \langle x, \mu_U, \gamma_U \rangle$, then by (4), $\mu_U(p) = 1$ and $\gamma_U(p) = 0$. It thus follows that $\mathcal{U} = \{U_p^n : p \in suppA$ and $n \geq m$ with $p(\frac{1}{n}, 1 - \frac{1}{n}) \leq A\}$ forms an I-fuzzy open cover of A such that for $U_p^n \in \mathcal{U}$, there exists $F_p^n \in \mathcal{F}$ with $cl(U_p^n)\ \overline{q}\ F_p^n$[by (2)]. Since A is an IFAC set in X, there exist finitely many members $U_{p_1}^{n_1}, U_{p_2}^{n_2}, ..., U_{p_k}^{n_k}$ of $\mathcal{U}$ such that $A \subseteq \bigcup_{i=1}^{k} cl(U_{p_i}^{n_i}) = cl(\bigcup_{i=1}^{k} U_{p_i}^{n_i}) = C$(say). Thus C is an I-fuzzy regular closed set containing A. Again for each $U_{p_i}^{n_i}$ of $\mathcal{U}$, there exists $F_{p_i}^{n_i}$ of $\mathcal{F}$ such that $cl(U_{p_i}^{n_i})\ \overline{q}\ F_{p_i}^{n_i}$ for each $i = 1, 2, ...k$. Thus $\bigcup_{i=1}^{k} cl(U_{p_i}^{n_i})\ \overline{q}\ (\bigcap_{i=1}^{k} F_{p_i}^{n_i}) \Rightarrow cl(\bigcup_{i=1}^{k} U_{p_i}^{n_i})\ \overline{q}\ (\bigcap_{i=1}^{k} F_{p_i}^{n_i}) \Rightarrow C\ \overline{q}\ (\bigcap_{i=1}^{k} F_{p_i}^{n_i})$. Hence C is an I-fuzzy regular closed set containing A such that $C\ \overline{q}\ (\bigcap_{i=1}^{k} F_{p_i}^{n_i})$, which contradicts our hypothesis.

Conversely let $\mathcal{B}$ be an I-fuzzy open filterbase in X such that $[\bigcap \{clB : B \in \mathcal{B}\}] \bigcap A = 0_\sim$. So $\mathcal{B}$ has no I-fuzzy cluster point in A and hence by Theorem 3.1, $\mathcal{B}$ has no I-fuzzy θ-cluster point contained in A. Then by hypothesis, there exists an I-fuzzy regular closed set F containing A such that $\bigcap \mathcal{B}_0\ \overline{q} F$, for some finite subcollection $\mathcal{B}_0$ of $\mathcal{B}$. Then $\bigcap \mathcal{B}_0\ \overline{q} A$ and consequently by Theorem 2.8, A is an IFAC set in X. □

Theorem 3.4. *Let A be an IFAC set in an IFTS (X, τ). Then every I-fuzzy filterbase $\mathcal{F}$ in A with the property that every finite intersection of members of $\mathcal{F}$ is q-coincident with at least one member of $\mathcal{F}$, has an I-fuzzy θ-cluster point in A.*

Proof. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an IFAC set in an IFTS X. If possible, let there exist an I-fuzzy filterbase $\mathcal{F}$ in A such that $A \bigcap [\bigcap cl_\theta F : F \in \mathcal{F}] = 0_\sim$. Then proceeding as in the above proof (necessity part) we arrive at an I-fuzzy open cover $\mathcal{U} = \{U_x^n : x \in suppA$ and $n \in \mathbb{N}$ with $x(\frac{1}{n}, 1 - \frac{1}{n}) \leq A\}$ of A such that for each U_x^n of $\mathcal{U}$, there exists $F_x^n \in \mathcal{F}$ with $cl(U_x^n)\ \overline{q}\ F_x^n$. Since A is an IFAC set in X, there exist $U_{x_1}^{n_1}, U_{x_2}^{n_2}, ..., U_{x_k}^{n_k}$ of $\mathcal{U}$ such that $A \subseteq \bigcup_{i=1}^{k} cl(U_{x_i}^{n_i})$. Then $F_{x_1}^{n_1} \bigcap F_{x_2}^{n_2} \bigcap ... \bigcap F_{x_k}^{n_k}\ \overline{q}\ A$ and hence $F_{x_1}^{n_1} \bigcap F_{x_2}^{n_2} \bigcap ... \bigcap F_{x_k}^{n_k}\ \overline{q}\ F$, for all $F \in \mathcal{F}$ (as $F \subseteq A, \forall F \in \mathcal{F}$), which contradicts our hypothesis. Hence $\mathcal{F}$ has an I-fuzzy θ-cluster point in A. □

Example 3.5. Let $X = \{a, b\}$ and $\tau = \{1_\sim, 0_\sim\} \cup \{B_n : n \in \mathbb{N}\} \cup \{\bigcup_{n \in \mathbb{N}} B_n\}$, where $B_1 = \langle x, (\frac{\mu_a}{0.3}, \frac{\mu_b}{0}), (\frac{\gamma_a}{0}, \frac{\gamma_b}{0.4}) \rangle$, $B_2 = \langle x, (\frac{\mu_a}{0.33}, \frac{\mu_b}{0}), (\frac{\gamma_a}{0}, \frac{\gamma_b}{0.04}) \rangle$, $B_3 = \langle x, (\frac{\mu_a}{0.333}, \frac{\mu_b}{0}), (\frac{\gamma_a}{0}, \frac{\gamma_b}{0.004}) \rangle$ and so on. Then (X, τ) is an IFTS. let $A = \langle x, (\frac{\mu_a}{0.4}, \frac{\mu_b}{0}), (\frac{\gamma_a}{0}, \frac{\gamma_b}{0}) \rangle$ be an IFS in X. Then clearly A is an IFAC set as each I-fuzzy open cover of A must include $1_\sim$ as a member of it. Now, we see that $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ is an I- fuzzy filterbase in A such that every finite intersection of members of $\mathcal{B}$ is q-coincident with at least one member of $\mathcal{B}$, since for any two members U, V of $\mathcal{B}$, $\mu_U(a) > \gamma_V(a)$ in X. Let $a(0.3, 0)$ be an IFP with support a. Here $a(0.3, 0) \leq A$. Now for each $B_i \in \mathcal{B}$, $a(0.3, 0)qB_i$, since $0.3 > \gamma_{B_i}(a)$, $\forall i \in \mathbb{N}$. so each member of $\mathcal{B}$ is an open q-nbd of $a(0.3, 0)$. Also, $\mu_{B_i}(a) > \gamma_{B_j}(a)$, $\forall i, j \in \mathbb{N}$. Thus B_iqB_j and hence clB_iqB_j for any $i, j \in \mathbb{N}$. This means $a(0.3, 0) \leq cl_\theta B_i, \forall i \in \mathbb{N}$. So, $a(0.3, 0) \leq \bigcap_{B \in \mathcal{B}} cl_\theta B$, i.e., $a(0.3, 0)$ is an I- fuzzy θ- cluster point of $\mathcal{B}$. This verified Theorem 3.4.

Theorem 3.6. *For any I-fuzzy open filterbase $\mathcal{F}$ in an IFAC space X, $A = \bigcap\{cl_\theta F : F \in \mathcal{F}\}$ is an IFAC set in X.*

Proof. Let $\mathcal{B}$ be an I-fuzzy open filterbase in an IFAC space X such that for each finite subcollection $\mathcal{B}_0$ of $\mathcal{B}$, $\bigcap\{B : B \in \mathcal{B}_0\}\ q\ A$. Let us take the collection $\mathcal{G} = \{B \cap F : B \in \mathcal{B}, F \in \mathcal{F}\}$. We first show that the members of $\mathcal{G}$ are non-null. Let us take any two members $(B_1 \cap F_1)$ and $(B_2 \cap F_2)$ from $\mathcal{G}$. By supposition, $(B_1 \cap B_2)\ q\ A \Rightarrow$ there exists an IFP $p(\alpha, 1-\alpha) \leq A$ such that $p(\alpha, 1-\alpha)q(B_1 \cap B_2)$. Now there exists $F_3 \in \mathcal{F}$ for which $F_3 \subseteq F_1 \cap F_2$. Since $p(\alpha, 1-\alpha) \leq cl_\theta F_3$ so $cl(B_1 \cap B_2)\ q\ F_3 \Rightarrow cl(B_1 \cap B_2)\ q\ (F_1 \cap F_2) \Rightarrow (B_1 \cap B_2)\ q\ (F_1 \cap F_2)$ (by Lemma 1.7). Hence $(B_1 \cap F_1) \cap (B_2 \cap F_2) \neq 0_\sim$.

Again $\mathcal{B}$ is an I-fuzzy filterbase so that for $B_1, B_2 \in \mathcal{B}$, there exists $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$. Thus $(B_3 \cap F_3) \subseteq (B_1 \cap B_2) \bigcap (F_1 \cap F_2) = (B_1 \cap F_1) \bigcap (B_2 \cap F_2)$ and $\mathcal{G}$ is an I-fuzzy open filterbase in X. Since X is IFAC, $\mathcal{G}$ has an I-fuzzy cluster point in X (by Theorem 2.7). Therefore $0_\sim \neq \bigcap\{cl(B \cap F) : B \cap F \in \mathcal{G}\} \subseteq [\bigcap\{clB : B \in \mathcal{B}\}] \bigcap [\bigcap\{clF : F \in \mathcal{F}\}]$ ($\because cl(A \cap B) \subseteq clA \cap clB$) $= [\bigcap\{clB : B \in \mathcal{B}\}] \bigcap [\bigcap\{cl_\theta F : F \in \mathcal{F}\}]$ ($\because F$ is open, $clF = cl_\theta F$) $= [\bigcap\{clB : B \in \mathcal{B}\}] \bigcap A$. Hence by Theorem 2.8, we can say that A is an IFAC set. □

Let us now verify the above theorem by means of an example as follows.

Example 3.7. Let $X = \{a, b\}$ and $\tau = \{1_\sim, 0_\sim\} \cup \{U_n, V_n : n \in \mathbb{N}\} \cup \{\bigcup_{n=1}^{\infty} U_n\}$, where $U_1 = \langle x, (\frac{a}{0.4}, \frac{b}{0.5}), (\frac{a}{0.3}, \frac{b}{0.4})\rangle$, $U_2 = \langle x, (\frac{a}{0.44}, \frac{b}{0.55}), (\frac{a}{0.03}, \frac{b}{0.04})\rangle$, $U_3 = \langle x, (\frac{a}{0.444}, \frac{b}{0.555}), (\frac{a}{0.003}, \frac{b}{0.004})\rangle$ and so on; and $V_1 = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.5})\rangle$, $V_2 = \langle x, (\frac{a}{0.03}, \frac{b}{0.04}), (\frac{a}{0.44}, \frac{b}{0.55})\rangle$, $V_3 = \langle x, (\frac{a}{0.003}, \frac{b}{0.004}), (\frac{a}{0.444}, \frac{b}{0.555})\rangle$ and so on. Then it is a routine work to check that (X, τ) is an IFTS. Also X is an IFAC space, since every I-fuzzy open cover of X must include $1_\sim$ as a member of it. Now, for any finite sub-collection, say $\{V_{n_1}, V_{n_2}, ..., V_{n_k}\}$ of $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$, $\bigcap_{i=1}^{k} V_{n_i} = V_{n_j}$, where $n_j = max(n_1, ..., n_k)$. Thus $\mathcal{V}$ is an I-fuzzy open filterbase in X. Now, $cl_\theta V_n = clV_n = U_n, \forall n \in \mathbb{N}$ [since $U_n = \overline{V_n}, \forall n \in \mathbb{N}$]. Thus $\cap\{clV_n : n \in \mathbb{N}\} = \cap\{U_n : n \in \mathbb{N}\} = U_1$, and U_1 is clearly an IFAC set in X, i.e., $\cap\{cl_\theta V_n : n \in \mathbb{N}\}$ is an IFAC set.

For the last theorem of this paper we need the following lemma:

Lemma 3.8. *For any I-fuzzy set A in an IFTS (X, τ), $cl_\theta A = \bigcap\{cl_\theta U : U \in \tau$ and $A \subseteq U\}$.*

Proof. Clearly L.H.S$\subseteq$ R.H.S. On the other hand, if possible, let $p(a, b) \leq \bigcap\{cl_\theta U : U \in \tau$ and $A \subseteq U\}$ but $p(a, b) \nleq cl_\theta A$. Then there exists an I-fuzzy open q-nbd V of $p(a, b)$ such that $clV\ \overline{q}\ A$. Thus $A \subseteq (1 - clV)$. So $p(a, b) \leq cl_\theta(1 - clV)$ and hence $clV\ q\ (1 - clV)$ which is impossible. Thus our assumption is wrong and this completes the proof. □

Theorem 3.9. *For any I-fuzzy set A in an IFAC space X, $cl_\theta A$ is an IFAC set in X.*

Proof. Let $\mathcal{F}$ be any I-fuzzy filterbase in an IFAC space X with the property that for every finitely many members $F_1, F_2, ..., F_n$ of $\mathcal{F}$ and for each I-fuzzy regular closed set C containing $cl_\theta A$, $(\bigcap_{i=1}^{n} F_i)\ q\ C$ ... (1)

By Theorem 3.3, it is sufficient to show that $\mathcal{F}$ has an I-fuzzy θ-cluster point in $cl_\theta A$. Let U be any I-fuzzy open set in X containing A. Then clU is an I-fuzzy regular closed set (by Proposition 2.4 (i)) such that $cl_\theta A \subseteq cl_\theta U = clU$ (by Theorem 3.1). Then by (1) above, for any finite subcollection $F_1, F_2, ..., F_n$ of $\mathcal{F}$, $(\bigcap_{i=1}^{n} F_i)\ q\ clU$ ... (2)

Let W be any I-fuzzy open set in X containing $\bigcap_{i=1}^{n} F_i$ for any finite subcollection $\{F_1, F_2, ..., F_n\}$ of $\mathcal{F}$. We claim that for any I-fuzzy open set U in X containing A, $W\ q\ U$ ... (3)

If not, then let $W\ \overline{q}\ U$. Then $U \subseteq \overline{W} \Rightarrow clU \subseteq \overline{W} \subseteq \overline{(\bigcap_{i=1}^{n} F_i)}$. Then $clU\ \overline{q}\ (\bigcap_{i=1}^{n} F_i)$ contradicting (2).

Now let $\mathcal{F}^*$ denote the set of all finite intersections of members of $\mathcal{F}$; $\mathcal{U}$ denote the set of all I-fuzzy open sets, each of which contains some member of $\mathcal{F}^*$ and $\mathcal{V}$ denote the set of all I-fuzzy open sets containing A.

We consider the collection $\mathcal{G} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Then for any two members $(U_1 \cap V_1), (U_2 \cap V_2)$ of $\mathcal{G}$, we have $U_1 \cap U_2 \in \mathcal{U}$ and $V_1 \cap V_2 \in \mathcal{V}$, Then by (3), $(U_1 \cap U_2)\ q\ (V_1 \cap V_2)$ i.e., $[(U_1 \cap V_1) \bigcap (U_2 \cap V_2)] \neq 0_\sim$ ($\because UqV \Rightarrow U \cap V \neq 0_\sim$). $\Rightarrow [(U_1 \cap U_2) \bigcap (V_1 \cap V_2)] \neq 0_\sim$. Also $(U_1 \cap U_2) \bigcap (V_1 \cap V_2) \in \mathcal{G}$ and hence $\mathcal{G}$ is an I-fuzzy open filterbase in X. Since X is IFAC, $\bigcap \{clG : G \in \mathcal{G}\} \neq 0_\sim$. ... (4)

Now let for each $F \in \mathcal{F}$, $\mathcal{U}_F$ denote the set of all I-fuzzy open sets, each containing F. Then $\mathcal{U}_F \subseteq \mathcal{U}$ for all $F \in \mathcal{F}$. Then for each $F \in \mathcal{F}$, we have $F \in \mathcal{F}^*$ and by Lemma 3.8, $\bigcap\{clU : U \in \mathcal{U}\} \subseteq \bigcap\{clU : U \in \mathcal{U}_F\} = cl_\theta F$. $\Rightarrow \bigcap\{clU : U \in \mathcal{U}\} \subseteq \bigcap\{cl_\theta F : F \in \mathcal{F}\}$... (5)

Also we have $\bigcap\{clG : G \in \mathcal{G}\} \subseteq \bigcap\{clU : U \in \mathcal{U}\}$ and $\bigcap\{clG : G \in \mathcal{G}\} \subseteq \bigcap\{clV : V \in \mathcal{V}\}$. Thus by (4), we get $0_\sim \neq \bigcap\{clG : G \in \mathcal{G}\} \subseteq [\bigcap\{clU : U \in \mathcal{U}\}] \bigcap \{\bigcap clV : V \in \mathcal{V}\}$ $\subseteq [\bigcap\{cl_\theta F : F \in \mathcal{F}\}] \bigcap [\bigcap\{clV : V \in \mathcal{V}\}]$ (by (5)) $= \bigcap\{cl_\theta F : F \in \mathcal{F}\} \bigcap cl_\theta A$ (by Theorem 3.1 and Lemma 3.8). Thus $\mathcal{F}$ has an I-fuzzy θ-cluster point in $cl_\theta A$ and hence by Theorem 3.3, it follows that $cl_\theta A$ is an IFAC set in X. □

Example 3.10. It follows from Example 2.2. that $1_\sim = cl(A) \subseteq cl_\theta(A) \Rightarrow cl_\theta(A) = 1_\sim$ and thus $cl_\theta(A)$ is also IFAC (as noted Example 2.2), which verifies the above Theorem.

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