

2023

MATHEMATICS — HONOURS

Paper : DSE-B-2.1

(Point Set Topology)

Full Marks : 65

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.*Throughout the question, \mathbb{R} and \mathbb{N} denote respectively the set of real numbers and the set of natural numbers.1. Answer all multiple choice questions. For each question, 1 mark for choosing correct option and 1 mark for justification. 2×10

- (a) Let (X, τ) be a cofinite topological space, where X is an uncountable set. Then which of the following is false?
- Each point of X is the intersection of all of its neighbourhoods in X .
 - No two open sets in X are disjoint.
 - $\tau \subseteq \tau'$, where τ' denotes the co-countable topology on X .
 - There exists a metric on X which generates the topology τ on X .
- (b) Let (X, τ) be a topological space and A be a proper non-empty subset of X such that $\text{int}(X-A) = \phi$, (where $\text{int } B$ denotes the interior of any subset B in X). Then which of the following is false?
- A is dense in X .
 - Every non-empty open set in X intersects A .
 - The only closed set in X containing A is X .
 - The derived set of A is an empty set.
- (c) Let \mathbb{R} be the set of all real numbers with usual topology and $K = \left\{ \frac{1}{n} : n = 1, 2, \dots \right\}$. Then K is
- open in \mathbb{R} .
 - closed in \mathbb{R} .
 - both open and closed in \mathbb{R} .
 - neither open nor closed in \mathbb{R} .
- (d) The closure of the set $A = \left\{ 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots \right\}$ with respect to the usual topology on the set of real numbers \mathbb{R} is given by
- $A \cup \{1\}$.
 - $A \cup \{2\}$.
 - $A \cup \{\phi\}$.
 - $A \cup \{3\}$.

Please Turn Over

- (e) Let $f: (\mathbb{R}, \tau_l) \rightarrow (\mathbb{R}, \tau_u)$ be defined as $f(x) = x, \forall x \in \mathbb{R}$, where τ_l, τ_u are the lower limit topology and the usual topology on \mathbb{R} respectively, then
- f is not a continuous map.
 - f is an open map.
 - f is neither continuous nor an open map.
 - f is continuous but not an open map.
- (f) Let (X, τ) be a co-countable space, where X is an uncountable set. Then which of the following is true?
- (X, τ) is a first countable space.
 - (X, τ) is a Hausdorff space.
 - There exists a convergent sequence in X whose limit is not unique.
 - A sequence $\{x_n\}$ in X is convergent if and only if there is some positive integer m such that for all $n \geq m, x_n = \text{constant}$.
- (g) Let $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ be a topology on $X = \{a, b, c\}$. Then X is
- compact and Hausdorff.
 - compact but not Hausdorff.
 - only Hausdorff.
 - T_1 .
- (h) A continuous function f from an infinite connected space (X, τ) to a discrete two point space $\{0, 1\}$
- must be constant.
 - must be non-constant.
 - is not closed.
 - is not open.
- (i) Let (X, τ) be an uncountable compact space and (\mathbb{R}, τ_u) be the space of real numbers with the usual topology. Then which of the following is false?
- There exists a continuous map $f: X \rightarrow \mathbb{R}$ which is unbounded.
 - A map $f: X \rightarrow \mathbb{R}$ is continuous $\Rightarrow f: X \rightarrow \mathbb{R}$ is a closed map.
 - If $f: X \rightarrow \mathbb{R}$ is a continuous map then $f(X)$ is closed in \mathbb{R} .
 - A map $f: X \rightarrow \mathbb{R}$ is continuous and $A \in \tau$ implies $f(X \setminus A)$ is compact in \mathbb{R} .
- (j) Let $X = [0, 1) \cup [2, 3]$ be the subspace of the topological space \mathbb{R} with the usual topology and $f: X \rightarrow \mathbb{R}$ be a map given by $f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 2 & \text{if } 2 \leq x \leq 3. \end{cases}$
- Then which of the following is true?
- f is open and continuous.
 - f is open but not continuous.
 - f is not open but continuous.
 - f is neither open nor continuous.

Unit - 1

(Marks : 20)

Answer *any four* questions.

2. (a) Let \mathbb{N} be the set of natural numbers and $A_n = \{1, 2, 3, \dots, n\}$, $n \in \mathbb{N}$. Then prove that the collector $\tau = \{A_n : n \in \mathbb{N}\} \cup \{\mathbb{N}, \emptyset\}$ is a topology on \mathbb{N} .
 (b) Find the derived set of $\{1\}$ in the above topological space. 3+2
3. (a) Prove that the lower limit topology τ_l and the upper limit topology τ_r are both strictly finer than the usual topology τ_u on the set of all real numbers \mathbb{R} .
 (b) Prove that the lower limit topology τ_l and the upper limit topology τ_r on \mathbb{R} are non-comparable but their intersection is the usual topology τ_u on \mathbb{R} . 2+3
4. Define topologically equivalent metrics on a non-empty set X . Prove that the space $(X, \tau(d))$, where $\tau(d)$ is the topology induced by a metric d on a non-empty set X is homeomorphic to the space $(X, \tau(d_1))$, where $\tau(d_1)$ is the topology induced by the metric d_1 on X , where d_1 is given by $d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$, for all $x, y \in X$. 1+4
5. (a) Suppose (X, τ) is a topological space and (Y, τ_Y) is the subspace of (X, τ) . Prove that, for a subset A of Y , $\bar{A}^Y = \bar{A} \cap Y$, where \bar{A}^Y denotes the closure of A in (Y, τ_Y) .
 (b) Find the boundary and interior of the set $\{(x, y) : x \in \mathbb{Q}\}$ in \mathbb{R}^2 endowed with the usual product topology. 3+2
6. Prove that in a topological space (X, τ)
 (a) the set $A \cup A'$ is the smallest closed subset containing A , where $A \subseteq X$ and A' is the derived set of A .
 (b) Prove or disprove : $\overline{A \cap B} = \bar{A} \cap \bar{B}$, where \bar{A} denotes the closure of A in (X, τ) . 3+2
7. (a) Let X be a non-empty set and $B = \{\{x\} : x \in X\}$. Then prove that B is a basis for a topology on X .
 (b) Give an example of a map from a topological space (X, τ) to another topological space (Y, τ') which is both open and closed but not continuous. 3+2
8. Let (X, τ) be the topological product of a family of topological spaces $\{(X_i, \tau_i) : i = 1, 2, \dots, n\}$ and $p_i : X \rightarrow X_i$ denote the i -th projection map $\forall i = 1, 2, \dots, n$. Then prove that
 (a) p_i is an open map $\forall i = 1, 2, \dots, n$.
 (b) the product topology τ is the smallest topology on X such that each projection map is continuous. 2+3

Please Turn Over

Unit - 2

(Marks : 10)

Answer *any two* questions.

9. (a) Give example of a topological space which is T_1 but not T_2 . Justify your answer.
 (b) Prove that a topological space (X, τ) is T_1 if and only if every neighbourhood of any limit point p of any set $A \subseteq X$ intersects A in countably infinite number of points. 2+3
10. Let X be an uncountable set and p be a fixed point X . Define $\tau = \{G \subseteq X : \text{either } p \notin G \text{ or if } p \in G \text{ then } X \setminus G \text{ is finite}\}$. Prove that (X, τ) is a topological space which is not first countable. 2+3
11. (a) Let (X, τ) be a topological space and \mathcal{B} a local base at $c \in X$. Prove that a sequence $\{x_n\}_n$ converges to $c \in X$ if and only if for every $B \in \mathcal{B}$, there exists a positive integer m such that for all $n \geq m$, $x_n \in B$.
 (b) Let $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ be an open, continuous surjective map, where X is first countable. Prove that Y is first countable. 3+2
12. (a) If (X_1, τ_1) and (X_2, τ_2) are two T_2 spaces, then prove that their product space (X, τ) is also a T_2 space.
 (b) Let $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ be a continuous map and Y be T_2 . Prove that the set $\{(x, f(x)) : x \in X\}$ is a closed set in $X \times Y$, where $X \times Y$ is endowed with the product topology. 3+2

Unit - 3

(Marks : 15)

Answer *any three* questions.

13. (a) Suppose (X, τ) is a topological space and $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ (Λ is an index set) is any family of closed subsets of X with the property that $\bigcap_{i=1}^n F_{\alpha_i} \neq \emptyset$ for any finite subfamily $\{F_{\alpha_i} : \alpha_i \in \Lambda, i=1, 2, \dots, n\}$ of \mathcal{F} . Prove that X is compact if and only if $\bigcap_{\alpha \in \Lambda} F_\alpha \neq \emptyset$.
 (b) Let (X, τ_1) be a T_2 space and (X, τ_2) be compact such that $\tau_1 \subseteq \tau_2$. Prove that $\tau_1 = \tau_2$. 3+2
14. (a) Prove that the set of real numbers \mathbb{R} with lower limit topology is disconnected.
 (b) Prove that a topological space containing a dense connected set is connected. 2+3
15. (a) Prove that a real valued continuous function f on a compact space (X, τ) attains its greatest value.
 (b) If K is a compact subset of a T_2 space X , then prove that K is a closed set in X . 2+3

16. (a) Prove that every closed and bounded interval of the real line \mathbb{R} (i.e., \mathbb{R} with the usual topology) is compact.
- (b) Prove that each component of a topological space is closed. 3+2
17. (a) If every continuous real valued function on a topological space (X, τ) takes on all values between any two values that it assumes then prove that (X, τ) is connected.
- (b) If A is a connected subset consisting of at least two points in a metric space (X, d) then prove that A is uncountable. 2+3
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