2024

MATHEMATICS — HONOURS

Paper : CC-5

(Theory of Real Functions)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$ denotes the set of real numbers.

Group - A

(Marks : 20)

1. Answer the following multiple choice questions each having only one correct option. Choose the correct option and justify. (1+1)×10

(a) $\lim_{x \to 1} \sin \frac{\pi[x]}{2}$ is equal to

(i) 0 (ii) 1

(iii) $\frac{\pi}{2}$ (iv) none of these.

([x] denotes the largest integer not exceeding x)

(b) If $\lim_{x \to 0} f(x) = \frac{2}{3}$, then $\lim_{n \to \infty} f\left(\frac{1}{n^2}\right)$ is equal to

(i) 0 (ii) $\frac{2}{3}$

(iii) $\frac{3}{2}$ (iv) $+\infty$.

(c) If $f : [0, 2] \to \mathbb{R}$ is defined by $f(x) = x - [x]$, then the set of points of discontinuity of f is

(i) $\{0, 2\}$ (ii) $\{0, 1, 2\}$

(iii) $\{1, 2\}$ (iv) $[0, 2]$.

(d) $f(x)=\begin{cases} \dfrac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

(i) has removable discontinuity at '0' (ii) has jump discontinuity at '0'

(iii) has oscillatory discontinuity at '0' (iv) is continuous at '0'.

(e) If $f(x)=\sin\frac{1}{x}$, $x \in (0, 1)$, then f is

(i) uniformly continuous on (0, 1) (ii) continuous but not uniformly continuous on (0, 1)

(iii) discontinuous on (0, 1) (iv) monotonic on (0, 1).

(f) Let $f(x)=\begin{cases} x\cos\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$. Then f is

(i) continuous but not derivable at '0' (ii) not continuous at '0'

(iii) derivable at '0' (iv) unbounded in every neighbourhood of '0'.

(g) $\lim\limits_{x\to 0+} \dfrac{x-\tan x}{x^3}$ is equal to

(i) $-\frac{1}{2}$ (ii) $\frac{1}{3}$

(iii) $\frac{1}{2}$ (iv) $-\frac{1}{3}$.

(h) If $f(x)=x^2$, $g(x)=x$ in $[-1, 1]$, then the value of

$\xi \in (-1, 1)$ satisfying $\dfrac{f(1)-f(-1)}{g(1)-g(-1)}=\dfrac{f'(\xi)}{g'(\xi)}$ is

(i) 1 (ii) 0

(iii) – 1 (iv) $\frac{1}{2}$.

(i) $f(x) = \begin{cases} 2x+3 & , \ x \geq 0 \\ -3x+1 & , \ x < 0 \end{cases}$ has

(i) local maximum at $x = 2$

(ii) local minimum at $x = -\frac{2}{5}$

(iii) local maximum at $x = -\frac{2}{5}$

(iv) neither a local maximum nor a local minimum at any point of $\mathbb{R}$.

(j) The set $\left\{x \in \mathbb{R} : e^{x} > 5\right\}$ is

(i) open (ii) closed

(iii) both open and closed (iv) neither open nor closed.

Group - B

(Marks : 25)

Answer ***any five*** questions.

2. (a) Let $S \subseteq \mathbb{R}$, 'a' be a limit point of S and $f : S \to \mathbb{R}$ be a function. If for every sequence $\{x_n\}$ of elements of $S - \{a\}$ converging to 'a', $\{f(x_n)\}$ converges to $\ell \in \mathbb{R}$, then prove that $\lim_{x \to a} f(x) = l$.

(b) Show that $\lim_{x \to 0} \frac{1}{x} \sin \frac{1}{x}$ does not exist. 3+2

3. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then prove that f is bounded on $[a, b]$ and attains its bounds in $[a, b]$. 3+2

4. Let $f : (a, b) \to \mathbb{R}$ be monotonically increasing on (a, b) and $c \in (a, b)$.

Prove that $\lim_{x \to c-} f(x)$ and $\lim_{x \to c+} f(x)$ exist and $\lim_{x \to c-} f(x) \leq \lim_{x \to c+} f(x)$. 4+1

5. Let $f : I \to \mathbb{R}$ be continuous on the interval I. Prove that $|f|$ is also continuous on I. Is the converse true? Justify your answer. 3+2

6. Let $f : I \to \mathbb{R}$ satisfy Lipschitz condition on the interval I. Prove that f is uniformly continuous on I. Is the converse true? Justify your answer. 3+2

7. Let I be a bounded interval and $f : I \to \mathbb{R}$ be uniformly continuous on I. Prove that f is bounded on I. Does the conclusion remain valid if uniformity condition is dropped? Give reason in support of your answer. 3+2

8. Let $f : [0, 2] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 4 - x & \text{if } 0 \le x < 1 \\ 4 & \text{if } x = 1 \\ 6 - x & \text{if } 1 < x \le 2. \end{cases}$

Is f piecewise continuous on [0, 2]? Justify your answer. Find the set of points of continuity of f. 4+1

9. (a) Prove that if f is continuous at 'a' and for every $\delta > 0$, there is a point $c_\delta \in (a - \delta, a + \delta)$ such that $f(c_\delta) = 0$, then $f(a) = 0$.

(b) Give an example of discontinuous function that satisfies the Intermediate Value Property on its domain. Justify your answer. 2+3

Group - C

(Marks : 20)

Answer ***any four*** questions.

10. (a) If $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ \sin x, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ show that $f(x)$ is derivable at $x = 0$ and $f'(0) = 1$.

(b) Give an example of a function which is differentiable everywhere but derived function is not continuous on its domain. 3+2

11. (a) Show that a derived function on an interval $[a, b]$ can not have a jump discontinuity on $[a, b]$.

(b) Does there exist a function ϕ such that $\phi'(x) = f(x)$ on $[0, 2]$, where $f(x) = x - [x]$? Justify your answer. 3+2

12. If $f'(c)$ exists, prove that $\lim\limits_{h \to 0} \dfrac{f(c + h) - f(c - h)}{2h} = f'(c)$.

Show by an example that the above limit may exist even if f is not derivable at 'c'. 3+2

13. (a) If f' exists in [0, 1], then show that $f(1) - f(0) = \dfrac{f'(x)}{2x}$ has at least one solution in [0, 1].

(b) If $\phi(x) = f(x) + f(1 - x)$, $x \in [0, 1]$ and $f''(x) < 0 \ \forall \ x \in [0, 1]$, show that ϕ is increasing on $\left[0, \frac{1}{2}\right]$ and decreasing on $\left[\frac{1}{2}, 1\right]$.

2+3

14. Expand $\log(1 + x)$ in powers of x as an infinite series and mention the region for validity of the expansion.

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15. (a) Evaluate $\lim_{x \to \infty} \left[x - \{(x - a_1)(x - a_2) \ldots (x - a_n)\}^{\frac{1}{n}} \right]$; $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

(b) Is Mean Value Theorem applicable to $f(x) = \begin{cases} x \sin \frac{1}{x} &, \ x \neq 0 \\ 0 &, \ x = 0 \end{cases}$ in [–1, 1]? Justify your answer.

3+2

16. Show that the conical tent of given capacity will require the least amount of canvas if its height is $\sqrt{2}$ times the radius of the base.

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