2024

MATHEMATICS — HONOURS

Paper : CC-7

(ODE and Multivariate Calculus – I)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$ denotes the set of real numbers.

Group - A

(Marks : 20)

1. Answer the following multiple-choice questions with only one correct option. Choose the correct option and justify : (1+1)×10

(a) The equation of the integral curve through the point (2, 3) corresponding to the differential equation $\frac{dy}{dx}=\frac{y}{x}$ is

(i) $3y = 2x$ (ii) $2y = 3x$

(iii) $y = 3x$ (iv) None of these.

(b) An integrating factor of the differential equation $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ is

(i) $\frac{1}{xy}$ (ii) $\frac{1}{x^2+y^2}$

(iii) $\frac{1}{x^2y^2}$ (iv) $\frac{1}{x^2-y^2}$.

(c) Which of the following differential equations is linear?

(i) $5\frac{dy}{dx}+x\sqrt{y}=5e^{4x}$ (ii) $\frac{dy}{dx}+5x^2\tan y=x$

(iii) $e^x\frac{dy}{dx}+y=e^{-x}$ (iv) $y\frac{dy}{dx}+5xy=\log_e x$.

(d) The Wronskian of the functions $y_1 = e^x \sin x$ and $y_2 = e^x \cos x$ is

(i) 0 (ii) $e^x \cos x$

(iii) $e^x \sin x$ (iv) $-e^{2x}$.

(e) The singular solution of the equation $p^2 + xp - y = 0$ is

(i) $y^2 = 4x$ (ii) $x^2 = -4y$

(iii) $x^2 + y^2 = xy$ (iv) $xy = 4$.

(f) Which of the following is correct for the linear differential equation

$$(3x+1)x\frac{d^2y}{dx^2} - (x+1)\frac{dy}{dx} + 3y = 0\ ?$$

(i) 0 is an irregular singular point (ii) -1 is an irregular singular point

(iii) -1 is a regular singular point (iv) no irregular singular point.

(g) The value of $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\sin\left(x^2+y^2\right)}{x^2+y^2}$ is

(i) 1 (ii) -1

(iii) 0 (iv) does not exist.

(h) The unit normal to the surface $x^2 + y^2 = z$ at point (1, 2, 5) is

(i) $\frac{2}{\sqrt{21}}\bar{i} - \frac{4}{\sqrt{21}}\bar{j} - \frac{\bar{k}}{\sqrt{21}}$ (ii) $\frac{-2}{\sqrt{21}}\bar{i} + \frac{4}{\sqrt{21}}\bar{j} - \frac{\bar{k}}{\sqrt{21}}$

(iii) $\frac{2}{\sqrt{21}}\bar{i} + \frac{4}{\sqrt{21}}\bar{j} - \frac{\bar{k}}{\sqrt{21}}$ (iv) $\frac{2}{\sqrt{21}}\bar{i} + \frac{4}{\sqrt{21}}\bar{j} - \frac{\bar{k}}{\sqrt{21}}$.

(i) If $z = \dfrac{x+y}{x-y}, x \neq y$, the value of $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}$ is

(i) $2\dfrac{x+y}{x-y}$ (ii) $2\dfrac{x+y}{(x-y)^2}$

(iii) $\dfrac{2}{x-y}$ (iv) $\dfrac{-2}{x-y}$.

(j) For the function $f(x, y) = x^2 - y^3 - x^2y + y$, the point $\left(0, \dfrac{1}{\sqrt{3}}\right)$ is

(i) not a critical point (ii) a saddle point

(iii) a point of local minimum (iv) a point of local maximum.

Group - B

(Marks : 30)

Answer ***any six*** questions.

2. (a) State the existence and uniqueness theorem for the initial value problem $\frac{dy}{dx}=f(x,y),\ y(x_0)=y_0$.

   (b) Solve the equation $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)$. 2+3

3. Find the value of the constant λ for which the differential equation $\left(2xe^y+3y^2\right)\frac{dy}{dx}+\left(3x^2+\lambda e^y\right)=0$ is exact. Hence solve the equation. 2+3

4. Solve : $x^2y-x^3\frac{dy}{dx}y=y^4\cos x$. 5

5. Reduce the equation $y^2(y-xp)=x^4p^2$ to Clairaut's form by the substitution $x=\frac{1}{u}$, $y=\frac{1}{v}$ and hence solve it. Find the singular solution, if it exists. 2+2+1

6. Find the solution of the differential equation $(D^2-2D+1)y=xe^x$ $\left(D\equiv\frac{d}{dx}\right)$ by the method of variation of parameters. 5

7. Solve the differential equation $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+4y=8x^2+3+2\cos 2x$ by the method of undetermined coefficients. 5

8. Find the general solution of the following Euler-Cauchy differential equation :
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x$$ 5

9. Find x and y from the system of equations :
$$\frac{dx}{dt}+4x+3y=t$$
$$\frac{dy}{dt}+2x+5y=e^t$$ 5

10. Determine the nature and stability of the critical point (0, 0) of the following system :

$$\frac{dx}{dt} = 3x + 2y$$
$$\frac{dy}{dt} = x + 2y$$

Also draw rough sketch of the corresponding phase portraits. 3+2

11. Solve the equation $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = 0$ about the ordinary point $x = 0$. 5

Group - C

(Marks : 15)

Answer ***any three*** questions.

12. (a) Show that the set $S = \{(x, y) \in \mathbb{R}^2 : 0 \le x < 1,\ 0 \le y < 1\}$ is neither open nor closed.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} y + x\sin\left(\frac{1}{y}\right), & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ but $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ does not exist. 2+3

13. Examine the existence of maxima or minima of the function $f(x, y) = x^2 + y^2 + (x + y + 1)^2$. 5

14. If $u = f(x^2 + 2yz,\ y^2 + 2zx)$, prove that

$$\left(y^2 - zx\right)\frac{\partial u}{\partial x} + \left(x^2 - yz\right)\frac{\partial u}{\partial y} + \left(z^2 - xy\right)\frac{\partial u}{\partial z} = 0.$$ 5

15. Find the maximum or minimum of the function $f(x, y) = xy$, subject to the condition $5x + y = 13$, using the method of Lagrange's Multipliers. 5

16. Find the directional derivative of the function $\phi(x, y, z) = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$. Also find the greatest rate of increase of ϕ. 3+2
