2024

MATHEMATICS — HONOURS

Paper : CC-6

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Choose the correct alternative with proper justification (**1 mark** for correct answer and **1 mark** for justification) : (1+1)×10

(a) If R is a ring with unity I and S is a subring or R, then

(i) S has unity I (ii) S has unity which may or may not be I

(iii) S may not have unity (iv) S has unity different from I.

(b) The number of idempotents in the ring $(\mathbb{Z}_{12}, +, \cdot)$ is

(i) 2 (ii) 3

(iii) 4 (iv) 6.

(c) Which of the following statements is not true?

(i) Every maximal ideal in a commutative ring with unity is a prime ideal.

(ii) Every prime ideal in a commutative ring with unity is a maximal ideal.

(iii) If P is a prime ideal of a commutative ring R with unity, then R/P is an integral domain.

(iv) If M is a maximal ideal in a commutative ring R with unity, then R/M is a field.

(d) The ideal $4\mathbb{Z} + 10\mathbb{Z}$ of $\mathbb{Z}$ is equal to

(i) $20\mathbb{Z}$ (ii) $2\mathbb{Z}$

(iii) $40\mathbb{Z}$ (iv) $10\mathbb{Z}$.

(e) Let R be a ring with unity 1_R and I be an ideal of R and I contains a unit of R. Then which of the following is true?

(i) I is a proper ideal of R (ii) $I = R$

(iii) $I = \{0_R\}$ (iv) $1_R \notin I$.

(f) Let $f: \mathbb{C} \to \mathbb{R}$ be a homomorphism such that $f(x+iy) = x$. Then Ker$f$ is

(i) $\{x : x \in \mathbb{R}\}$ (ii) $\{0\}$

(iii) $\{ix : x \in \mathbb{R}\}$ (iv) $\{x + iy \in \mathbb{C} : x \neq 0 \text{ and } y \neq 0\}$.

Please Turn Over

(g) A $n \times n$ matrix A is non-singular if and only if

(i) all the eigenvalues of A are distinct

(ii) $\det A = 0$

(iii) 0 is an eigenvalue of A

(iv) 0 is not an eigenvalue of A.

(h) Let V_1 and V_2 be finite dimensional subspaces of a vector space V. If $\dim(V_1) = 2$, $\dim(V_2) = 3$, $\dim(V_1 + V_2) = 4$, then $\dim(V_1 \cap V_2)$ is

(i) 1 (ii) 2

(iii) 3 (iv) 4.

(i) If $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined is by, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$; then Ker T is equal to

(i) $\{\alpha(1, 1, -1) : \alpha \in \mathbb{R}\}$ (ii) $\{\alpha(1, -1, 1) : \alpha \in \mathbb{R}\}$

(iii) $\{\alpha(-1, 1, 1) : \alpha \in \mathbb{R}\}$ (iv) $\{\alpha(1, 1, 1) : \alpha \in \mathbb{R}\}$.

(j) Which one of the following polynomial is satisfied by the matrix A where $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$?

(i) $x^3 + 5x^2 - 5x - 1$ (ii) $x^3 - 5x^2 + 5x + 1$

(iii) $x^3 - 5x^2 - 5x + 1$ (iv) $x^3 - 5x^2 + 5x - 1$.

Unit - I

2. Answer ***any five*** questions :

(a) (i) Prove that the characteristic of an integral domain is either zero or a prime number.

(ii) If R is a commutative ring of prime characteristic p, prove that $(a+b)^p = a^p + b^p$ for all $a, b \in R$. 3+2

(b) (i) Let a, b (with $b \neq 0$) be two elements of a field F with $(ab)^2 = ab^2 + bab - b^2$, prove that $a = 1$.

(ii) Prove that a field does not contain a divisor of zero. 3+2

(c) (i) Examine if S is a field, where $S = \{a + b\sqrt{2} : a, b \in Z\}$.

(ii) Show that the mapping $f : \mathbb{Z}[\sqrt{2}] \to M_2(\mathbb{Z})$ defined by $f(a + b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$ is a homomorphism of rings. Find Ker f. 2+(2+1)

(d) (i) Let R be a ring and $a \in R$. Show that $S = \{x \in R : ar = x \text{ for some } r \in R\}$ is a subring of R.

(ii) If a, b, c be any three elements of a ring be $(R, +, \cdot)$, show that $(a - b) - c = a - (b + c)$. 3+2

(e) If R is a commutative ring with identity and M be an ideal of R, show that R/M is a field if and only if M is a maximal ideal of R. 5

(f) Let P, Q be prime ideals of a commutative ring R. Prove that $P \cap Q$ is a prime ideal of R if and only if either $P \subseteq Q$ or $Q \subseteq P$. 5

(g) (i) Let R and R' be two rings and $\phi : R \to R'$ be a homomorphism. Then show that $\text{Ker } \phi$ is an ideal of R.

(ii) Let M be a maximal ideal of a commutative ring R with unity. Then show that M is a prime ideal of R. 3+2

(h) (i) Find the maximal ideals and prime ideals in the ring $\mathbb{Z}_6$.

(ii) A surjective ring homomorphism $f : \mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z}_5$ is defined by, $f(x) = ([x]_3, [x]_5)$. Find $x \in \mathbb{Z}$ such that, $f(x) = ([2]_3, [3]_5)$. Determine $\text{Ker} f$. 2+(2+1)

Unit - II

3. Answer ***any four*** questions :

(a) (i) Let V be a vector space of dimension n over a field F. Prove that any linearly independent set of n vectors of V is a basis of V.

(ii) If $\{\alpha_1, \alpha_2, \alpha_3\}$ be a basis of a real vector space V and

$\beta_1 = \alpha_1 + \alpha_3$, $\beta_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3$

$\beta_3 = \alpha_1 + 2\alpha_2 + 3\alpha_3$, prove that $\{\beta_1, \beta_2, \beta_3\}$ is also a basis of V. 3+2

(b) (i) Prove that a linearly independent set of vectors in a finite dimensional vector space V over a field F is either a basis of V, or it can be extended to a basis of V.

(ii) Extend the set $S = \{(1, 2, 1), (2, 1, 1)\}$ to a basis of $\mathbb{R}^3$. 3+2

(c) Let V and W be finite dimensional vector spaces of same dimension over a field F and $T : V \to W$ be a linear mapping. Then prove that T is an isomorphism if and only if T maps a basis of V to a basis of W. 5

(d) Let V and W be two vector spaces over a field F and $T : V \to W$ be a linear mapping. If the dimension of V is finite, show that $\dim V = \dim \mathit{ker}\, T + \dim \mathit{Im}\, T$. 5

(e) (i) Show that the eigenvalues of a Hermitian matrix are real.

(ii) λ is an eigenvalue of a real skew symmetric matrix. Prove that $\left|\frac{1-\lambda}{1+\lambda}\right| = 1$. 2+3

(f) The matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ relative to the ordered basis $\{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$ of $\mathbb{R}^3$ is $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$. Find the matrix of T relative to the ordered basis $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$. 5

(g) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Applying Cayley–Hamilton theorem to prove that $A^n - A^{n-2} = A^2 - I$, for all integers $n \geq 3$. Hence find A^{40}. 3+2