2022

MATHEMATICS — HONOURS

Paper : CC-11

(Probability and Statistics)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Each of the following questions has four possible answers of which exactly one is correct. Choose the correct alternative with proper justification (wherever applicable) : 2×10

(a) For any two events A and B, $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{1}{2}$. Then $P(A \mid B)$ is

(i) $\frac{1}{3}$ (ii) $\frac{1}{2}$

(iii) $\frac{1}{4}$ (iv) 1.

(b) The probability density function is $f(x) = \begin{cases} x, & \text{when } 0 < x < 1 \\ 2-x, & \text{when } 1 \le x < 2 \\ 0, & \text{elsewhere} \end{cases}$

then the value of $P\left(\frac{1}{2} < X < \frac{3}{2}\right)$ is

(i) $\frac{1}{4}$ (ii) $\frac{1}{2}$

(iii) $\frac{2}{3}$ (iv) $\frac{3}{4}$.

(c) If $E(X) = 3$, $E(Y) = 5$ and $\text{cov}(X, Y) = 2$, then $E(XY)$ is

(i) 17 (ii) 13

(iii) 39 (iv) 32.

(d) If $f(x,y)=\begin{cases} K(1-x^2-y^2), & 0<x^2+y^2<1 \\ 0, & \text{otherwise} \end{cases}$ is a probability density function of two random variable X and Y, then

(i) $K=-\frac{1}{\pi}$ (ii) $K=-\frac{2}{\pi}$

(iii) $K=\frac{2}{\pi}$ (iv) $K=\frac{1}{\pi}$.

(e) The moment generating function of a random variable X uniformly distributed over the interval (−1, 1) is given by

(i) $\frac{\sinh t}{t}$, $(t \neq 0)$ (ii) $\frac{\cosh t}{t}$, $(t \neq 0)$

(iii) $\frac{\tanh t}{t}$, $(t \neq 0)$ (iv) $\frac{\sin t}{2t}$, $(t \neq 0)$

(f) If the mean and standard deviation of 10 observations $x_1, x_2, \ldots, x_{10}$ are 2 and 3 respectively, then the mean of $(x_1+1)^2, (x_2+1)^2, \ldots\ldots, (x_{10}+1)^2$ is

(i) 9 (ii) 18

(iii) 15.5 (iv) 18.2.

(g) Let X be a binomial random variable with parameter n and p, where n is a positive integer and $0<p<1$.

If $\alpha = P\left(|X-np| \geq \sqrt{n}\right)$, then which of the following statements holds true for all n and p ?

(i) $0 \leq \alpha \leq \frac{1}{4}$ (ii) $\frac{1}{4} < \alpha \leq \frac{1}{2}$

(iii) $\frac{1}{2} < \alpha < \frac{3}{4}$ (iv) $\frac{3}{4} \leq \alpha \leq 1$.

(h) A coin is tossed 4 times and p is the probability of getting head in a single trial. Let S be the number of head(s) obtained. It is decided to test $H_0: p=\frac{1}{2}$ against $H_1: p \neq \frac{1}{2}$, using the decision rule. Reject H_0 if S is 0 or 4. Then the Type-I error is

(i) $\frac{1}{2}$ (ii) $\frac{1}{4}$

(iii) $\frac{1}{8}$ (iv) $\frac{1}{16}$.

(i) Two random variable X and Y have least square regression lines $3x + 2y = 13$ and $6x + y = 1$. The correlation coefficient r_{xy} has value

(i) $\frac{1}{3}$　　(ii) $-\frac{1}{2}$

(iii) $\frac{1}{2}$　　(iv) $\frac{1}{4}$.

(j) In case of fitting a bivariate data exponentially to $y = ab^x$, we may fit a straight line if

(i) $\log y$ is plotted against x　　(ii) e^y is plotted against x

(iii) y is plotted against x　　(iv) y is plotted against $\log x$.

Unit - 1

Answer *any two* questions.

2. (a) From the numbers 1, 2, 3, ..., 101 three are chosen at random. Find the probability these are in arithmetic progression.

(b) Construct three events which are pairwise independent but they are not mutually independent. 3+2

3. In the equation $x^2 + 2x - q = 0$, q is a random variable uniformly distributed over the interval (0, 2). Find the distribution function of the larger root. 5

4. If X is a Normal (m, σ) variate, then prove that $\mu_{2k+2} = \sigma^2 \mu_{2k} + \sigma^3 \frac{d\mu_{2k}}{d\sigma}$, where μ_k denotes the k-th central moment. Hence find μ_4. 4+1

Unit - 2

Answer *any two* questions.

5. The joint density function of the random variables X, Y is given by

$$f(x, y) = 2 \quad (0 < x < 1,\ 0 < y < x)$$

Find the marginal and conditional density functions. Compute $P(\frac{1}{4} < X < \frac{3}{4} \mid Y = \frac{1}{2})$ 5

6. If (X, Y) is a two-dimensional random variable and ρ is the correlation coefficient between them, then show that $[E(XY)]^2 \leq E(X^2)\,E(Y^2)$. Hence deduce $-1 \leq \rho \leq 1$. 3+2

7. Two points are independently chosen at random in the interval (0,1). Find the probability that the distance between them is less than a fixed number k $(0 < k < 1)$. 5

Unit - 3

Answer *any one* question.

8. Show that central limit theorem for equal component implies law of large numbers for equal components. 5

9. Show by Tchebycheff's inequality that in 2000 throws with a coin the probability of getting heads between 900 and 1100 is at least $\frac{19}{20}$. 5

Unit - 4

Answer *any two* questions.

10. Prove that for a random sample $(x_1, x_2, \ldots, x_n)$ drawn from a normal (m, σ) population the sample mean $\overline{X}$ follows normal $\left(m, \frac{\sigma}{\sqrt{n}}\right)$ distribution. 5

11. Define an unbiased and consistent estimator T of a statistical attribute θ. Show that T^2 may not be unbiased when T is so but T^2 will be consistent when T is so. (2+3)

12. The population of scores of 10-years children in a test is known to have a standard deviation 5.2. If a random sample of size 20 shows a mean of 16.9, find 95% confidence limits for the mean score of the population, assuming that the population is normal. 5

$$\left[\text{Given : } \frac{1}{\sqrt{2\pi}} \int_{1.96}^{\infty} e^{-x^2/2} dx = 0.025\right]$$

13. Prove that the maximum likelyhood estimate of the parameter α of the population having density function $f(x) = \frac{2(\alpha - x)}{3\alpha^2}$ $(0 < x < \alpha)$ for a sample x_1 of unit size is $2x_1$. Test whether the estimate is biased or not. 3+2

Unit - 5

Answer *any two* questions.

14. 11 sample values taken at random from the measurement of fuel efficiency of cars are 14.2, 12.2, 13.1, 11.2, 12.4, 13.3, 14.4, 12.6, 11.4, 13.5, 14.6. Is it reasonable to believe that the standard deviation of the population of measured values is greater than 1.1? Assume that the population is normal and use 5% level of significance.

Given that P $(\chi^2 > 18.307) = 0.05$ where χ^2 is random variable having chi-square distribution with 10 degrees of freedom. 5

15. In a township the milk consumption of the families is assumed to be exponentially distributed with parameter λ. The hypothesis $H_0 : \lambda = 5$ is rejected in favour of $H_1 : \lambda = 10$ if a family, selected at random, consumes 15 units or more. Find Type-I error and power of the test. 2+3

16. A bivariate sample of size 11 gave the results $\bar{x} = 7$, $S_x = 2$, $\bar{y} = 9$, $S_y = 4$ and $r = 0.5$. It was later found that one pair of the sample values ($x = 7$, $y = 9$) was inaccurate and was rejected. How would the original value of r be affected by the rejection? 5

(All symbols have their usual meanings)

17. Fit the following data to a parabolic curve : 5

x	2	2.8	3	5
y	17	22	40	70