2022

MATHEMATICS — HONOURS

Paper : DSE-A-1.2

[Bio-Mathematics]

(Unit - 3)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group - A

(Marks : 20)

1. Answer the following multiple-choice questions with only one correct option. Choose the correct option with proper justification : (1+1)×10

(a) For exponential growth $\frac{dN}{dt} = rN$ with growth rate $r(>0)$,

the population doubling time will be

(i) $\ln 2$ (ii) $r \ln 2$

(iii) $\frac{1}{r}\ln 2$ (iv) $\frac{1}{r^2}\ln 2$.

(b) For the single species continuous system $\frac{dx}{dt} = 0.3x\left(1-\frac{x}{10}\right)$,

(i) trivial equilibrium is stable but non-trivial equilibrium is unstable.

(ii) trivial equilibrium is unstable but non-trivial equilibrium is asymptotically stable.

(iii) both the equilibria are unstable.

(iv) both the equilibria are stable.

(c) For the harvesting model $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - h$, where $r, K, h > 0$, the unique equilibrium exists if

(i) $h < \frac{rK}{4}$ (ii) $h > \frac{rK}{4}$

(iii) $h = \frac{rK}{4}$ (iv) none of these.

(d) The system $\frac{dx}{dt} = \mu + x^2$, $x, \mu \in \mathbb{R}$, has

(i) pitchfork bifurcation (ii) saddle node bifurcation

(iii) transcritical bifurcation (iv) none of these.

(e) The non-negative steady states of the equations

$$\frac{dx}{dt} = x^2 - y^2$$
$$\frac{dy}{dt} = x(1-y)$$

are

(i) (0, 0); (1, 0) (ii) (0, 0); (1, 1)

(iii) (1, 0); (1, 1) (iv) none of these.

(f) For the system

$$\frac{dx}{dt} = -2y$$
$$\frac{dy}{dt} = x,$$

the steady state (0, 0) is

(i) stable spiral (ii) saddle

(iii) centre (iv) none of these.

(g) In the following chemostat model

$$\frac{dx}{dt} = \left(K(c) - D\right)x$$
$$\frac{dc}{dt} = D\left(c_0 - c\right) - \frac{1}{p}K(c)x,$$

where the symbols have their usual meanings, the equilibrium point $(0, c_0)$ is stable if

(i) $D < K(c_0)$ (ii) $D > 2K(c_0)$

(iii) $D > K(c_0)$ (iv) $2D < K(c_0)$.

(h) In the following Lotka-Volterra predator-prey model

$$\frac{dx}{dt} = x(\alpha - \beta y)$$
$$\frac{dy}{dt} = y(-\gamma + \delta x),$$

$\alpha, \beta, \gamma, \delta$ are positive parameters, the equilibrium point (0, 0) is

(i) stable node (ii) saddle point

(iii) stable focus (iv) unstable focus.

(i) The discrete logistic type model $x_{n+1} = rx_n(1 - x_n), r > 0,$ the trivial steady state is stable if

(i) $r > 1$ (ii) $0 < r < 1$

(iii) $1 < r < 2$ (iv) none of these.

(j) The steady state $x^* = 5$ of the system $x_{n+1} = x_n e^{5 - x_n}$ is

(i) asymptotically stable (ii) stable but not asymptotically stable

(iii) unstable (iv) none of these.

Group - B

Unit - I

(Marks : 15)

Answer ***any one*** question.

2. (a) Write down the continuous-time Logistic model equation for a single-species population and explain the variables and parameters involved. Find the analytical solution of the model equation. Discuss the behaviour of population size as time approaches infinity. How does Malthus model become a special case of Logistic model?

(b) What is an equilibrium point or steady state of $\frac{dN}{dt} = f(N)$? When is a steady state called asymptotically stable? Suppose that N^* is a steady state of $\frac{dN}{dt} = f(N)$, and $f(N)$ is a continuously differentiable function with $f'(N^*) \neq 0$. Prove that N^* is asymptotically stable if $f'(N^*) < 0$ and unstable if $f'(N^*) > 0$.

Hence, discuss the nature of stability of the steady states of the continuous-time Logistic model for a single-species population.

(2+2+1+1)+(1+2+3+3)

3. (a) What are the defects of the Malthus model? Write down the Gompertz model explaining the variables and parameters involved. Find the steady states and discuss their stability.

(b) Show that for the single natural fish population harvest model

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - qEN,$$

at $qE = r$, there is a transcritical bifurcation where E is the fishing effort and q measures catchability. Draw the bifurcation diagram.

(c) What do you understand by a pitchfork bifurcation? (2+2+2)+(4+2)+3

Unit - II

(Marks : 20)

Answer ***any two*** questions.

4. Consider the following modified Lotka-Volterra model :

$$\frac{dX}{dT} = rX\left(1-\frac{X}{K}\right) - bXY,$$
$$\frac{dY}{dT} = -dY + cXY,$$

where r, K, b, c, d are positive parameters.

(a) Using the substitutions $x = \frac{X}{K}, y = \frac{b}{r}Y$ and $t = rT$, reduce the above system in the following dimensionless form :

$$\frac{dx}{dt} = x(1-x-y)$$
$$\frac{dy}{dt} = \beta y(x-\alpha)$$

where α, β are the new parameters to be determined.

(b) Find all the steady states of the dimensionless system and discuss their stability. 3+(3+4)

5. (a) What do you mean by the predator functional response in a predator– prey system? Write down the different Holling type functional responses and draw the corresponding response curves.

(b) Discuss the stability of the steady states of the following predator-prey system :

$$\frac{dx}{dt} = x\left(1-\frac{x}{30}\right) - \frac{xy}{x+10}$$
$$\frac{dy}{dt} = y\left(\frac{x}{x+10} - \frac{1}{3}\right)$$

(1+3)+6

6. (a) What is meant by a limit cycle for a two-dimensional system? Show that the following system

$$\frac{dx}{dt} = y + \frac{x}{\sqrt{x^2+y^2}}\left(1-x^2-y^2\right)$$
$$\frac{dy}{dt} = -x + \frac{y}{\sqrt{x^2+y^2}}\left(1-x^2-y^2\right)$$

has a stable limit cycle.

(b) State Dulac criterion for a two-dimensional autonomous system. Using this show that the following system

$$\frac{dx}{dt} = x(\alpha - ax - by)$$
$$\frac{dy}{dt} = y(\beta - cx - dy)$$

where a, d are positive constants and α, β, b, c are any constants, has no closed orbits in the positive quadrant in $\mathbb{R}^2$. (1+4)+(2+3)

7. (a) Why is an epidemic model called compartmental model?

(b) Consider the compartmental model

$$\frac{dS}{dt} = -\lambda SI + \alpha R, \quad \frac{dI}{dt} = \lambda SI - \mu I, \quad \frac{dR}{dt} = \mu I - \alpha R,$$

where the parameters $\lambda, \mu, \alpha > 0$ are to be explained by you and $S(0)$, $I(0)$, $R(0)$ are initial values of the susceptible, infected and recovered populations respectively.

(i) Show that at any time t the total number of population is constant.

(ii) Reduce the system into two-dimensional system involving S and I population only.

(iii) Calculate the basic reproduction number R_0 and show that the epidemic can grow if $R_0 > 1$.

(iv) Discuss the stability of each equilibria of the reduced system. 1+(1+1+1+2+4)

Unit - III

(Marks : 10)

Answer *any one* question.

8. (a) Draw the Cobweb diagrams of the one-dimensional difference equation $x_{n+1} = rx_n$ for different values of the parameter r.

(b) When is a steady state of the one-dimensional system $x_{n+1} = f(x_n)$ said to be asymptotically stable? Discuss the stability of the steady states of the following one-dimensional model :

$$x_{n+1} = \frac{\lambda K\, x_n}{K + (\lambda - 1)x_n},$$

where λ and K are positive parameters. 5+5

9. (a) Let (x^*, y^*) be a steady state of the two-dimensional discrete–time system :

$$x_{n+1} = f(x_n, y_n),$$
$$y_{n+1} = g(x_n, y_n).$$

Linearise the system about (x^*, y^*) and hence state the conditions for stability of (x^*, y^*).

(b) Show that the coexistence equilibrium point of Nicholson–Bailey model

$$H_{n+1} = bH_n\, e^{-aP_n}$$
$$P_{n+1} = cH_n\left(1 - e^{-aP_n}\right)$$

is always unstable. 5+5

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